A ONE-DIMENSIONAL KINETIC MODEL OF PLASMA DYNAMICS WITH

A HYPERBOLIC FIELD

by

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Presented to the Faculty of the Honors College of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

HONORS BACHELOR OF ARTS IN MATHEMATICS

THE UNIVERSITY OF TEXAS AT ARLINGTON

April 2010

ACKNOWLEDGMENTS

ABSTRACT

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Publication No._____

Charles Nguyen, B.A.

The University of Texas at Arlington, 2010

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Motivated by the fundamental model of a collisionless plasma, the Vlasov-Maxwell (VM) system, we consider a related nonlinear system of partial differential equations posed in one space and one momentum dimension. As little is known concerning the regularity properties of solutions to the non-relativistic version of the (VM) equations, we study a simplified system, denoted by (SVM), that does not incorporate relativistic velocity corrections and prove the local-in-time existence of classical solutions to the Cauchy (or initial-value) problem. Uniqueness of solutions to (SVM) is also shown using *a priori* estimates of the particle density and magnetic field in addition to Gronwall's Inequality, a standard tool in the mathematical analysis of partial differential equations. Our method of proof entails defining a sequence of approximate solutions and showing that they converge in the appropriate sense to solutions of (SVM). As the limit of this sequence is not initially known, we show that the approximating sequence of functions and their derivatives are uniformly bounded and uniformly Cauchy, thereby implying their convergence and smoothness properties. The theorem presented within represents an initial step in the mathematical analysis of (SVM), ensuring that the physical problem has a solid foundation and is properly posed. These results also pave the way for future projects involving stability and computation of solutions to (SVM).

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INTRODUCTION AND BACKGROUND

A plasma is a partially or completely ionized gas. Nearly all (approximately 99.9%) of the matter in the universe exists in the state of plasma, as opposed to a solid, fluid, or a gaseous state. Such a form of matter occurs if the velocity of a material achieves an enormous magnitude, perhaps a sizable fraction of the speed of light. Hence all matter, if heated to a significantly great temperature, will reside in a plasma state. In terms of practical use, plasmas are of great interest to the energy, aeronautical, and aerospace industries among others, as they are used in the production of electronics, (plasma) engines, and lasers, as well as, in harnessing the power of nuclear energy. When a plasma is of low density or the time scales of interest are sufficiently small, it is deemed to be "collisionless", as collisions between particles become relatively infrequent. Many examples of collisionless plasmas occur in nature, including the solar wind, galactic nebulae, the Van Allen radiations belts, and comet tails.

The fundamental partial differential equations (PDEs) which describe the time evolution of a collisionless plasma are given by the Vlasov-Maxwell system:

$$\text{(VM)} \quad \begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0 \\[2mm] \rho(t, x) = \displaystyle\int f(t, x, v) \, dv, \quad j(t, x) = \int v f(t, x, v) \, dv \\[2mm] \partial_t E = \nabla \times B - j, \qquad \nabla \cdot E = \rho \\[2mm] \partial_t B = -\nabla \times E, \qquad \nabla \cdot B = 0. \end{cases}$$

Here, f represents the density of (positively-charged) ions in the plasma, while ρ and j are the charge and current density, and E and B represent electric and magnetic fields generated by the charge and current. The independent variables, $t > 0$ and $x, v \in \mathbb{R}^3$ represent time, position, and velocity, respectively, and physical constants, such as the charge and mass of particles, as well as, the speed of light, have been normalized to one. In the presence of large velocities, relativistic corrections become important and the corresponding system to consider is the relativistic analogue

of (VM), denoted by (RVM) and constructed by replacing v with

$$\hat{v} = \frac{v}{\sqrt{1 + |v|^2}}$$

in the first equation of (VM), called the Vlasov equation, and in the integrand of the current j. General references on the kinetic equations of plasma dynamics, such as (VM) and (RVM), include [6] and [11]. Over the past twenty-five years significant progress has been made in the analysis of (RVM), specifically, the global existence of weak solutions [4] and the determination of conditions which ensure global existence of classical solutions (originally discovered in [7], and later in [2] and [8]) for the Cauchy problem. Additionally, a wide array of information has been discovered regarding the electrostatic versions of both (VM) and (RVM) - the Vlasov-Poisson and relativistic Vlasov-Poisson systems, respectively. These models do not include magnetic effects within their formulation, and the electric field is given by an elliptic, rather than a hyperbolic, partial differential equation. This simplification has led to a great deal of progress concerning the electrostatic systems, including theorems regarding global existence, stability, and long-time behavior of solutions. Independent of these advances, many of the most basic existence and regularity questions remain unsolved for (VM). The main difficulty which arises is the loss of hyperbolicity of the system due to the possibility that particle velocities v may travel as fast as the propagation of signals from the electric and magnetic fields, which do so at the speed of light $c = 1$.

Often a remedy to the lack of progress on such a problem is to reduce the dimensionality of the system. Unfortunately, posing the problem in one-dimension (i.e., $x, v \in \mathbb{R}$) eliminates the relevance of the magnetic field as the Maxwell system decouples, yielding the one-dimensional Vlasov-Poisson system:

(VP)
$$\begin{cases} \partial_t f + v \partial_x f + E \partial_v f = 0 \\ \qquad \partial_x E = \int f \, dv. \end{cases}$$

The lowest-dimensional reduction which includes magnetic effects is the so-called "one-and-one-

half-dimensional" system, constructed by taking $x \in \mathbb{R}$ but $v \in \mathbb{R}^2$. Surprisingly, the question of classical regularity remains open even in this case. Thus, in order to study this question, but keep the problem posed in a one-dimensional setting, we consider the following nonlinear system of hyperbolic PDEs:

(SVM)
$$
\begin{cases}
\partial_t f + v \partial_x f + B \partial_v f = 0 \\
\partial_t B + \partial_x B = - \displaystyle\int f \, dv.
\end{cases}
$$

Since the field equation in (SVM) is hyperbolic, we denote it with a magnetic field variable B, as opposed to the electric field E of (VP). Notice that these equations retain the main difficulty of (VM), namely the interaction between characteristic particle velocities v and constant field velocities $c = 1$. The system (SVM) is supplemented by given initial conditions

(IC)
$$
f(0, x, v) = f_0(x, v), \quad B(0, x) = B_0(x).
$$

The main goal of the present work, then, is to determine whether there exist unique, smooth functions f and B which satisfy (SVM) and (IC) on some time interval, for any given smooth functions f_0 an B_0. To our knowledge, this is the first such study of these kinetic equations, formed from a system of hyperbolic conservation laws coupled by a non-local field dependence on the particle densities. As such, the properties of solutions to (SVM) may also be of interest to mathematicians studying scalar, hyperbolic conservation laws in a two-dimensional phase space. We also mention the work [1] as it investigates the stability properties of well-known steady state solutions to both (SVM) and (VP) augmented by a neutralizing particle density, the so-called BGK waves. Additionally, a work discussing computational methods for (SVM) is forthcoming in [3]. To conclude the introductory chapter, we will describe notation that will be utilized throughout and state the main goals and results of the thesis.

<u>1.1 Notation</u>

Before we can get into the mathematical details of the problem's formulation or solution, we must fix some notation that is standard to the current field of research, but not necessarily to a general mathematical audience. Thus, we define the following mathematical objects.

1. $C(\mathbb{R})$ is the set of continuous functions having domain $\mathbb{R}$.

2. For every $k \in \mathbb{N}$, $C^k(\mathbb{R})$ is the set of continuous functions which are k times continuously differentiable and defined on $\mathbb{R}$.

3. The *support* of a function is the set of values in its domain for which the function is non-zero.

4. For every $k \in \mathbb{N}$, $C_c^k(\mathbb{R})$ is the set of functions belonging to $C^k(\mathbb{R})$ for which the support of the function is a compact subset of $\mathbb{R}$.

5. For any $u \in C(\mathbb{R})$, we define $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$.

<u>1.2 Research Goal</u>

The major research goal of the thesis is to prove in a mathematically rigorous manner the existence and uniqueness of solutions to the density and magnetic field equations (SVM) for a collisionless plasma. This will be done in subsequent chapters by defining sequences of approximate solutions f^n and B^n which satisfy a linear system of partial differential equations and proving that these sequences and their derivatives converge to solutions of the nonlinear system (SVM), namely f and B. In order to show convergence, we must prove that the sequences are Cauchy as we do not know their limits *a priori*. Additionally, we will need uniform bounds on the terms of the sequences in order to establish the Cauchy property for the sequences and their derivatives. When combining all of this work, we arrive at the following theorem, the proof of which is contained in Chapters 2 through 5:

Theorem 1.1 (Existence of classical solutions)**.** *Let $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$ be given. Then, there exists $T > 0$ and a unique classical solution*

$$f \in C^1([0,T) \times \mathbb{R}^2), \quad B \in C^1([0,T) \times \mathbb{R})$$

to (SVM) satisfying the initial conditions (IC). Moreover, if we denote the maximal lifespan of the solution by T^* then for $T^* < \infty$ we must have

$$\limsup_{t \to T^*} \|\partial_x f(t)\|_\infty = \infty.$$

The structure of this work proceeds as follows. In the next chapter, we will begin the proof of the main theorem by showing uniqueness of solutions to (SVM) and describing the main ideas of the procedure by which we may prove that solutions to (SVM) exist. In Chapter 3 bounds on the particle density f and magnetic field B are obtained and the sequence of iterations described in Chapter 2 are shown to be Cauchy. Chapter 4 is devoted to similar arguments, but for derivatives of these functions. Finally, in Chapter 5 the convergence of this iterative scheme is shown and the properties of the resulting limits are deduced. Conclusions are stated in the final chapter. Since we are interested in classical solutions, we will also assume $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C(\mathbb{R})$ for the entirety of the paper. Throughout the paper the value $C > 0$ will denote a generic constant that may change from line to line. When necessary, we will specifically identify the quantities upon which C may depend.

UNIQUENESS AND SUCCESSIVE APPROXIMATIONS OF SOLUTIONS

In this chapter we show the uniqueness of the solution, namely that the solution to (SVM) is one of a kind. This is followed by a general outline for proving the existence of solutions using a sequence of approximating functions.

2.1 Uniqueness of Solutions

To begin, we will first prove that solutions to (SVM) and (IC) are unique, i.e. there is at most one distinct solution. This will be done by assuming that two solutions exist and then proving that they must, in fact, be equal. Hence, if there were more than one such solution, all of them would need to be the same. To do this, we will use a well-known lemma due to Gronwall:

Lemma 2.1 (Gronwall's Inequality). *Let $\zeta(t) \geq 0$ be an integrable function on $[0, T]$ satisfying*

$$\zeta(t) \leq C_1 \int_0^t \zeta(s)ds + C_2$$

for any $t \in [0, T]$ and some $C_1, C_2 \geq 0$. Then for all $t \in [0, T]$, ζ satisfies the bound

$$\zeta(t) \leq C_2 e^{C_1 t}.$$

This result can be found in [5] and is used in the following uniqueness proof.

Proof of Uniqueness: Let us first suppose that $(f^{(1)}, B^{(1)})$ and $(f^{(2)}, B^{(2)})$ are two solutions to the system (SVM) which satisfy (IC). Also, for every $t > 0$ and $x, v \in \mathbb{R}$ define the difference of these solutions

$$f(t, x, v) = f^{(1)}(t, x, v) - f^{(2)}(t, x, v)$$

$$B(t, x) = B^{(1)}(t, x) - B^{(2)}(t, x).$$

Then, we subtract the first equation of (SVM) for $f^{(2)}$ from that for $f^{(1)}$ to find

$$
\begin{aligned}
0 &= \partial_t f + v \partial_x f + B^{(1)} \partial_v f^{(1)} - B^{(2)} \partial_v f^{(2)} \\
&= \partial_t f + v \partial_x f + B^{(1)} \partial_v f^{(1)} - B^{(1)} \partial_v f^{(2)} + B^{(1)} \partial_v f^{(2)} - B^{(2)} \partial_v f^{(2)} \\
&= \partial_t f + v \partial_x f + B^{(1)} \partial_v f - B \partial_v f^{(2)}
\end{aligned}
$$

so that by rearranging terms this becomes

$$
\partial_t f + v \partial_x f + B^{(1)} \partial_v f = B \partial_v f^{(2)}.
$$

The left side of this equation can be expressed as a derivative along characteristic curves as

$$
\frac{d}{ds} f\big(s, X^{(1)}(s), V^{(1)}(s)\big) = \Big(B \partial_v f^{(2)} \Big)\big(s, X^{(1)}(s), V^{(1)}(s)\big)
$$

where the curves $X^{(1)}(s)$ and $V^{(1)}(s)$ are defined by the system of characteristic ordinary differential equations

(2.1)
$$
\begin{cases}
\dfrac{\partial X^{(1)}}{\partial s} = V^{(1)}(s, t, x, v), \\[2mm]
\dfrac{\partial V^{(1)}}{\partial s} = B^{(1)}(s, X^{(1)}(s, t, x, v)), \\[2mm]
X^{(1)}(t, t, x, v) = x, \\[2mm]
V^{(1)}(t, t, x, v) = v.
\end{cases}
$$

Here, we have abbreviated $X^{(1)}(s, t, x, v)$ by $X^{(1)}(s)$ and similarly for $V^{(1)}(s)$. Now, integrating both sides of the above equation with respect to s, we find

$$
f(t, x, v) - f\big(0, X^{(1)}(0), V^{(1)}(0)\big) = \int_0^t \big(B \partial_v f^{(2)} \big)\big(s, X^{(1)}(s), V^{(1)}(s)\big) ds
$$

and since both solutions satisfy the same initial condition (IC), this implies $f(0, x, v) \equiv 0$. There-

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fore, the equality simplifies to

$$f(t, x, v) = \int_0^t \left(B \partial_v f^{(2)}\right)\left(s, X^{(1)}(s), V^{(1)}(s)\right) ds.$$

Since $f^{(2)}$ is a solution, we know $\|\partial_v f^{(2)}(s)\|_\infty$ is bounded for $s \in [0, t]$ and we can bound the right side to find

(2.2)
$$\|f(t)\|_\infty \leq C \int_0^t \|B(s)\|_\infty \, ds.$$

Now, by subtracting the $B^{(2)}$ equation from the $B^{(1)}$ equation, we arrive at

$$\partial_t B + \partial_x B = -\int f \, dv$$

which we can write as a derivative along curves with slope one as

$$\frac{d}{ds} B(s, x - t + s) = -\int f(s, x - t + s, v) \, dv.$$

Integrating in s and using (IC) to conclude that $B(0, x) \equiv 0$, this becomes

$$B(t, x) = -\int_0^t \int f(s, x - t + s, v) \, dv ds.$$

Since these are solutions, the velocity support of f is controlled (see Chapter 3 for more details) and we can bound B by

(2.3)
$$\|B(t)\|_\infty \leq C\|f(t)\|_\infty$$

where C depends upon the existence time, the support of f_0, and the size of B_0. Finally, combining (2.2) and (2.3) yields

$$\|B(t)\|_\infty \leq C \int_0^t \|B(s)\|_\infty ds$$

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By Gronwall's Inequality (Lemma 2.1), we deduce $||B(t)||_\infty \leq 0$ which implies that $B(t, x) = 0$ for every $t > 0$, $x \in \mathbb{R}$. Similarly, using (2.2) we see that $f(t, x, v) = 0$ for every $t > 0, x, v \in \mathbb{R}$. Finally, this implies that $f^{(1)} \equiv f^{(2)}$ and $B^{(1)} \equiv B^{(2)}$, and hence there can be at most one such solution. $\qquad\square$

Now that we know there is at most one solution, we concentrate on showing that the problem (SVM) with (IC) does, in fact, possess a solution, namely the question of existence.

2.2 The Method of Successive Approximations

The outline of our proof of existence generally follows the structure of [9] and [10]. We begin with the general setup of the argument, which utilizes the method of successive approximations. Hence, we define an iterative sequence of solutions to linear partial differential equations and show that it must converge to a solution of the nonlinear system (SVM). To this end, we recall $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$ and define

$$f^0(t, x, v) = f_0(x, v),$$
$$B^0(t, x) = B_0(x).$$

Additionally, for every $n \in \mathbb{N}$, define $f^n \in C^1([0, T] \times \mathbb{R}^2)$ and $B^n \in C^1([0, T] \times \mathbb{R})$ by solving the linear initial-value problems

(2.4)
$$\begin{cases} \partial_t f^n + v\partial_x f^n + B^{n-1}\partial_v f^n = 0, \\ f^n(0, x, v) = f_0(x, v), \end{cases}$$

and

(2.5)
$$\begin{cases} \partial_t B^n + \partial_x B^n = -\int f^n dv, \\ B^n(0, x) = B_0(x), \end{cases}$$

respectively. Notice that if $f^n \to f$ and $B^n \to B$ as $n \to \infty$ in the appropriate sense then f and B

will satisfy (SVM) and (IC). We further define the sequence of functions which track the velocity support of f^n, namely

$$P^n(t) = \sup\{|v| : f^n(s, x, v) \neq 0, \ s \in [0, t], x \in \mathbb{R}\}.$$

In subsequent chapters we will rigorously prove that these sequences of approximating functions f^n and B^n must converge to limits and that these limits must then satisfy (SVM).

THE PARTICLE DENSITY AND MAGNETIC FIELD

In this chapter we construct bounds on the particle density and magnetic field and show that they satisfy the Cauchy property. This will be helpful in Chapter 5 to ultimately prove that solutions to (SVM) exist.

3.1 *A Priori* Bounds on the Density, Velocity Support, and Field

We begin this section by bounding the particle density f^n uniformly for $n \in \mathbb{N}$. Since the Vlasov equation is first-order, this requires the method of characteristics. For every $n \in \mathbb{N}$ define the characteristic curves $X^n(s, t, x, v)$ and $V^n(s, t, x, v)$ by

$$(3.1) \qquad \begin{cases} \dfrac{\partial X^n}{\partial s} = V^n(s, t, x, v), \\[2mm] \dfrac{\partial V^n}{\partial s} = B^{n-1}(s, X^n(s, t, x, v)), \\[2mm] X^n(t, t, x, v) = x, \\[2mm] V^n(t, t, x, v) = v. \end{cases}$$

Often, the (t, x, v) dependence of these curves will be suppressed so, for example, $X^n(s, t, x, v)$ will be denoted by $X^n(s)$ for brevity, as in the previous chapter for $X^{(1)}$. Then, the Vlasov equation can be expressed as a derivative along the characteristic curves by

$$\begin{aligned} \frac{d}{ds}\Big(f^n(s, X^n(s), V^n(s)) \Big) &= \partial_t f^n(s, X^n(s), V^n(s)) + \frac{\partial X^n}{\partial s} \partial_x f^n(s, X^n(s), V^n(s)) \\ &\quad + \frac{\partial V^n}{\partial s} \partial_v f^n(s, X^n(s), V^n(s)) \\ &= \partial_t f^n(s, X^n(s), V^n(s)) + V^n(s) \partial_x f^n(s, X^n(s), V^n(s)) \\ &\quad + B^{n-1}(s, X^n(s)) \partial_v f^n(s, X^n(s), V^n(s)) \\ &= 0 \end{aligned}$$

Then, this quantity is constant in s, so we find

$$f^n(t, X^n(t, t, x, v), V^n(t, t, x, v)) = f^n(0, X^n(0, t, x, v), V^n(0, t, x, v))$$

and using the initial condition with (3.1), for every $n \in \mathbb{N}$

(3.2) $$f^n(t, x, v) = f_0(X^n(0, t, x, v), V^n(0, t, x, v)).$$

Finally, after taking the supremum we find for every $t > 0$,

$$\|f^n(t)\|_\infty = \|f_0\|_\infty$$

and the particle densities are bounded in terms of their initial values.

In order to uniformly bound the sequence B^n, we utilize estimates on the velocity support of f^n. First, we use the method of characteristics again, but now to solve for B^n in terms of f^n. We write the differential equation for B^n as a derivative along the space-time curves $(s, x - t + s)$ so that

$$\frac{d}{ds}\left(B^n(s, x - t + s)\right) = \partial_t B^n(s, x - t + s) + \partial_x B^n(s, x - t + s) = \int f^n(s, x - t + s, v)\, dv.$$

Hence, integrating along characteristics, we arrive at

(3.3) $$B^n(t, x) = B_0(x - t) + \int_0^t \int f^n(s, x - t + s, v)\, dv\, ds.$$

Using the bounded data, compact velocity support, and uniform bound on the particle density yields

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$$
\begin{aligned}
|B^n(t,x)| \;&\leq\; C + \int_0^t \int_{\{|v|:f^n \neq 0\}} f^n(s, x - t + s, v)\, dv\, ds \\
&\leq\; C\left(1 + \int_0^t \|f^n(s)\|_\infty P^n(s)\, ds \right) \\
&\leq\; C\left(1 + \int_0^t P^n(s)\, ds \right).
\end{aligned}
$$

Thus, for every $n \in \mathbb{N}$ we have the bound

(3.4)
$$
\|B^n(t)\|_\infty \leq C\left(1 + \int_0^t P^n(s)\, ds \right).
$$

To bound the velocity support in terms of the field, we express the solution of the Vlasov equation in terms of the associated characteristics. From (3.2) it follows that

$$
\sup_{x,v} |V^n(t, 0, x, v)| \leq P^n(t)
$$

for every $t \geq 0$. The reader may consult [6] for more details regarding this point. From the velocity characteristic equation (3.1), we can integrate to find

$$
V^n(t, 0, x, v) = v + \int_0^t B^{n-1}(\tau, X^n(\tau))\, d\tau.
$$

Using the definition of P^n, this yields

$$
|V^n(t)| \leq P^n(0) + \int_0^t \|B^{n-1}(\tau)\|_\infty\, d\tau
$$

and taking the supremum over characteristics along which $f^n \neq 0$, we find

$$
P^n(t) \leq P^n(0) + \int_0^t \|B^{n-1}(\tau)\|_\infty\, d\tau.
$$

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We can now use (3.4) to arrive at a recursive bound for P^n, namely

$$P^n(t) \leq P^n(0) + C \int_0^t \left(1 + \int_0^s P^{n-1}(\tau)\, d\tau \right)\, ds.$$

Since f_0 has compact support, we know $P^n(0)$ is finite and constant in n, thus for every $n \in \mathbb{N}$ and on every bounded time interval $[0, T]$,

$$P^n(t) \leq C \left(1 + \int_0^t P^{n-1}(\tau)\, d\tau \right)$$

where C depends upon f_0 and the existence time. Using this recursive relation and the boundedness of $P^0(\tau) = P^0(0)$, we immediately deduce

$$P^n(t) \leq C \left(1 + \frac{t^n}{n!} \right) \leq C.$$

Thus, on any bounded time interval $[0, T]$ the function $P^n(t)$ is uniformly bounded and from (3.4) so is $\|B^n(t)\|_\infty$.

3.2 The Cauchy Property of the Density and Field

To show that the sequences f^n and B^n are Cauchy, we directly estimate the differences of terms of the sequences. Let $n, m \in \mathbb{N}$ be given and define the functions

$$f^{n,m}(t, x, v) = f^n(t, x, v) - f^m(t, x, v)$$

and

$$B^{n,m}(t, x) = B^n(t, x) - B^m(t, x).$$

Since the first equation of (SVM)

$$\partial_t f^n + v \partial_x f^n + B^{n-1} \partial_v f^n = 0$$

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holds for any $n \in \mathbb{N}$, we subtract the f^m equation from that for f^n to find

$$
\begin{aligned}
0 &= \partial_t f^{n,m} + v \partial_x f^{n,m} + B^{n-1} \partial_v f^n - B^{m-1} \partial_v f^m \\
&= \partial_t f^{n,m} + v \partial_x f^{n,m} + B^{n-1} \partial_v f^n - B^{n-1} \partial_v f^m + B^{n-1} \partial_v f^m - B^{m-1} \partial_v f^m \\
&= \partial_t f^{n,m} + v \partial_x f^{n,m} + B^{n-1} \partial_v f^{n,m} + B^{n-1,m-1} \partial_v f^m
\end{aligned}
$$

so that by rearranging terms this becomes

$$
\partial_t f^{n,m} + v \partial_x f^{n,m} + B^{n-1} \partial_v f^{n,m} = -B^{n-1,m-1} \partial_v f^m.
$$

The left side of this equation can be expressed as a derivative along characteristic curves in (3.1) as

$$
\frac{d}{ds} f^{n,m}\big(s, X^{n-1}(s), V^{n-1}(s)\big) = -\Big(B^{n-1,m-1} \partial_v f^m\Big)\big(s, X^{n-1}(s), V^{n-1}(s)\big).
$$

Now, integrating both sides with respect to s, we find

$$
f^{n,m}(t, x, v) - f^{n,m}\big(0, X^{n-1}(0), V^{n-1}(0)\big) = \int_0^t \big(B^{n-1,m-1} \partial_v f^m\big)\big(s, X^{n-1}(s), V^{n-1}(s)\big) ds
$$

and since both f^n and f^m satisfy the same initial condition (2.4), this implies $f^{n,m}(0, x, v) \equiv 0$. Therefore, the equality simplifies to

$$
f^{n,m}(t, x, v) = \int_0^t \big(B^{n-1,m-1} \partial_v f^m\big)\big(s, X^{n-1}(s), V^{n-1}(s)\big) ds.
$$

We know $\|\partial_v f^m(s)\|_\infty$ is bounded (from Chapter 2) so we can bound the right side to find

(3.5) $$ \|f^{n,m}(t)\|_\infty \le C \int_0^t \|B^{n-1,m-1}(s)\|_\infty \, ds. $$

Now, since the second equation of (SVM) must also hold for all $n \in \mathbb{N}$, we subtract the B^m

equation from that for B^n and arrive at

$$\partial_t B^{n,m} + \partial_x B^{n,m} = -\int f^{n,m}\ dv$$

which we can write as a derivative along curves with slope one as

$$\frac{d}{ds} B^{n,m}(s, x - t + s) = -\int f^{n,m}(s, x - t + s, v)\ dv.$$

Integrating in s and using the initial conditions (2.5) to conclude that $B^{n,m}(0, x) \equiv 0$, this becomes

$$B^{n,m}(t, x) = -\int_0^t \int f^{n,m}(s, x - t + s, v)\ dvds.$$

Since the velocity support of f, denoted by P^n, is uniformly bounded from Section 3.1, we can bound $B^{n,m}$ by

(3.6) $$\|B^{n,m}(t)\|_\infty \leq C \|f^{n,m}(t)\|_\infty$$

where C depends upon the time of existence. Finally, combining (3.5) and (3.6) yields

$$\|B^{n,m}(t)\|_\infty \leq C \int_0^t \|B^{n-1,m-1}(s)\|_\infty ds$$

for any $n, m \in \mathbb{N}$. Now consider $m = n - 1$ so that the previous equation becomes

$$\|B^{n,n-1}(t)\|_\infty \leq C \int_0^t \|B^{n-1,n-2}(s)\|_\infty ds.$$

Using this recursive relation we deduce

$$\|B^{n,n-1}(t)\|_\infty \leq \frac{C \|B^{1,0}(t)\|_\infty t^{n-1}}{(n-1)!}.$$

Since $\|B^1(t)\|_\infty$ and $\|B^0(t)\|_\infty$ are bounded from Section 3.1, we have

$$\|B^{n,m}(t)\|_\infty \leq \sum_{k=m+1}^{n} \|B^{k,k-1}(t)\|_\infty \leq C \sum_{k=m+1}^{n} \frac{t^{k-1}}{(k-1)!} \to 0$$

as $n, m \to \infty$ for every $t > 0$. Therefore, $B^n(t, x)$ is uniformly Cauchy for all $t > 0$ and $x \in \mathbb{R}$. Similarly, using (3.5) we see that $\|f^{n,m}(t)\|_\infty \to 0$ as $n \to \infty$ for every $t > 0$ and it follows that $f^n(t, x, v)$ is uniformly Cauchy for every $t > 0, x, v \in \mathbb{R}$. This useful property will be utilized in Chapter 5 as it implies that the sequences f^n and B^n must be uniformly convergent.

DERIVATIVES OF THE PARTICLE DENSITY AND MAGNETIC FIELD

In this chapter we construct bounds on the derivatives of the particle density and magnetic field and prove that they satisfy the Cauchy property, which in turn will show that the sequence of derivatives must converge uniformly.

4.1 *A Priori* Bounds on Derivatives

Now we focus on bounding derivatives, sketching the proof for x-derivatives with t-derivatives and v-derivatives following similarly. From the definition of the iterates we can differentiate (3.3) with respect to x so that

$$(4.1) \qquad \partial_x B^n(t, x) = B_0'(x - t) - \int_0^t \int \partial_x f^n(\tau, x - t + \tau, v) \, dv \, d\tau.$$

Using the conditions on the initial data B_0 and the bound on the velocity support P^n from Chapter 3, we arrive at

$$\|\partial_x B^n(t)\|_\infty \leq C \left(1 + \int_0^t \|\partial_x f^n(\tau)\|_\infty \, d\tau \right)$$

and thus

$$(4.2) \qquad \|\partial_x B^n(t)\|_\infty \leq C \left(1 + t \sup_{\tau \in [0,t]} \|\partial_x f^n(\tau)\|_\infty \right)$$

for every $n \in \mathbb{N}$. This tells us that we need to estimate derivatives of f^n as well. Differentiating the Vlasov equation in v yields

$$\left(\partial_t + v \partial_x + B^{n-1} \partial_v \right) \partial_v f^n = -\partial_x f^n$$

and upon integrating along characteristics we find

$$(4.3) \qquad \partial_v f^n(t, x, v) = (\partial_v f_0)(X^n(0), V^n(0)) - \int_0^t (\partial_x f^n)(s, X^n(s), V^n(s)) \, ds.$$

Taking supremums in x and v tells us that $\partial_v f^n$ is bounded if we can bound $\partial_x f^n$, that is

(4.4)
$$\|\partial_v f^n(t)\|_\infty \leq \|\partial_v f_0\|_\infty + \int_0^t \|\partial_x f^n(s)\|_\infty \, ds.$$

The estimate for $\partial_x f^n$ is derived similarly. Differentiating with respect to x in the Vlasov equation, we find

$$\left(\partial_t + v\partial_x + B^{n-1}\partial_v \right)\partial_x f^n = -\partial_x B^{n-1}\partial_v f^n$$

and integrating as before along characteristics yields

(4.5) $$\partial_x f^n(t, x, v) = (\partial_x f_0)(X(0), V(0)) - \int_0^t (\partial_x B^{n-1}\partial_v f^n)(s, X^n(s), V^n(s)) \, ds.$$

Taking the supremum on the right side, inserting the estimate on $\|\partial_v f^n(s)\|_\infty$ above, and using the bounded data gives us

$$
\begin{aligned}
|\partial_x f^n(t, x, v)| &\leq \|\partial_x f_0\|_\infty + \int_0^t \|\partial_x B^{n-1}(s)\|_\infty \|\partial_v f^n(s)\|_\infty \, ds \\
&\leq \|\partial_x f_0\|_\infty + \int_0^t \|\partial_x B^{n-1}(s)\|_\infty \left(C + \int_0^s \|\partial_x f^n(\tau)\|_\infty \, d\tau \right) ds \\
&\leq \|\partial_x f_0\|_\infty + C \int_0^t \|\partial_x B^{n-1}(s)\|_\infty \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f^n(\tau)\|_\infty \right) ds.
\end{aligned}
$$

Finally, this yields

(4.6) $$\|\partial_x f^n(t)\|_\infty \leq \|\partial_x f_0\|_\infty + C \int_0^t \|\partial_x B^{n-1}(s)\|_\infty \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f^n(\tau)\|_\infty \right) ds.$$

We then use the previous inequality (4.2) in (4.6) to find

$$\|\partial_x f^n(t)\|_\infty \leq C \left[1 + \int_0^t \left(1 + s \sup_{[0,s]} \|\partial_x f^{n-1}(\tau)\|_\infty \right) \left(1 + s \sup_{[0,s]} \|\partial_x f^n(\tau)\|_\infty \right) ds \right].$$

Let

$$F^n(s) = \max_{1 \le k \le n} \sup_{\tau \in [0,s]} \|\partial_x f^k(\tau)\|_\infty.$$

With this, the above inequality becomes

(4.7)
$$F^n(t) \le C \left(1 + \int_0^t [1 + sF^n(s)]^2 \, ds \right).$$

Hence, by induction $F^n(t) \le F(t)$ for every $n \in \mathbb{N}$, $t \in [0, T_0)$, where $F(t)$ is the maximal solution of the integral equation corresponding to (4.7), namely

(4.8)
$$F(t) = C \left(1 + \int_0^t [1 + sF(s)]^2 \, ds \right).$$

Here, $T_0 > 0$ is the maximal existence time of this solution and the constant C does not change from (4.7) to (4.8). Notice that $F(t)$ is guaranteed to exist on the time interval $[0, T_0)$ with $T_0 > 0$ determined by f_0 and B_0 only (through the constant C). This yields a uniform bound on $\|\partial_x f^n(t)\|_\infty$ on $[0, T]$ for every $n \in \mathbb{N}$ and $T < T_0$. Furthermore, it implies a uniform bound on $\|\partial_v f^n(t)\|_\infty$ on $[0, T]$ by (4.4). Additionally, $\|\partial_x B^n(t)\|_\infty$ is uniformly bounded on the same time interval by (4.2). Hence, derivatives of the iterates are locally bounded in time and uniformly so for $n \in \mathbb{N}$.

To conclude this section, we use the newly discovered boundedness of derivatives to show that the characteristics given by (3.1) are uniformly Cauchy. First, we let $m, n \in \mathbb{N}$ be given and define

$$X^{n,m}(s) = X^n(s) - X^m(s)$$

with the analogous definition for $V^{n,m}(s)$. Upon integrating the ODEs of (3.1) and subtracting the equations for X^m from those of X^n, we find

$$X^{n,m}(s) = - \int_s^t V^{n,m}(\tau) \, d\tau$$

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and thus

$$\text{(4.9)} \qquad \|X^{n,m}(s)\|_\infty \le \int_s^t \|V^{n,m}(\tau)\|_\infty d\tau.$$

Doing the same for the $V^n(s)$ equations, we use the Mean Value Theorem to find

$$
\begin{aligned}
|V^{n,m}(s)| &= \left| \int_s^t \Big(B^n(\tau, X^n(\tau)) - B^m(\tau, X^m(\tau)) \Big) d\tau \right| \\
&= \left| \int_s^t \Big(B^n(\tau, X^n(\tau)) - B^n(\tau, X^m(\tau)) + B^n(\tau, X^m(\tau)) - B^m(\tau, X^m(\tau)) \Big) d\tau \right| \\
&\le \int_s^t \Big(\|\partial_x B^n(\tau)\|_\infty |X^n(\tau) - X^m(\tau)| + \|B^{n,m}(\tau)\|_\infty \Big) d\tau.
\end{aligned}
$$

Since field derivatives are now bounded, this implies

$$\text{(4.10)} \qquad \|V^{n,m}(s)\|_\infty \le C \int_s^t \Big(\|X^{n,m}(\tau)\|_\infty + \|B^{n,m}(\tau)\|_\infty \Big) d\tau.$$

Finally, we let

$$Z^{n,m}(s) = \|X^{n,m}(s)\|_\infty + \|V^{n,m}(s)\|_\infty$$

and add (4.9) and (4.10) together to find

$$Z^{n,m}(s) \le C \int_s^t Z^{n,m}(\tau) \, d\tau + C \int_s^t \|B^{n,m}(\tau)\|_\infty \, d\tau.$$

As before, we use Gronwall's Inequality to find

$$Z^{n,m}(s) \le C \int_s^t \|B^{n,m}(\tau)\|_\infty \, d\tau.$$

Since we know $\|B^{n,m}(s)\|_\infty \to 0$ uniformly, this implies that $Z^{n,m}(s)$ does so as well, and finally that X^n and V^n are uniformly Cauchy.

4.2 The Cauchy Property of Derivatives

In order to prove that the resulting limits of f^n and B^n are differentiable, we will show that the sequence of derivatives $\partial_{(t,x,v)} f^n$ and $\partial_{(t,x)} B^n$ are Cauchy as well. We begin by bounding $\|\partial_x B^{n,m}(t)\|_\infty$. Using the representation from (4.1) and subtracting the equation for $\partial_x B^m$ from that for $\partial_x B^n$ we obtain

$$\partial_x B^{n,m}(t,x) = -\int_0^t \int \partial_x f^{n,m}(\tau, x - t + \tau, v)\, dv\, ds.$$

Using the boundedness of the velocity support from Chapter 3, this relationship implies

$$(4.11) \qquad \|\partial_x B^{n,m}(t)\|_\infty \le C \int_0^t \|\partial_x f^{n,m}(s)\|_\infty\, ds.$$

Then, since $\|\partial_v f^{n,m}(0)\|_\infty \equiv 0$, we use the representation for $\partial_v f^n$ from (4.3) and subtract this equation for $\partial_v f^m$ from that of $\partial_v f^n$ to find

$$(4.12) \qquad \|\partial_v f^{n,m}(t)\|_\infty \le \int_0^t \|\partial_x f^{n.m}(s)\|_\infty\, ds.$$

Similarly, since $\|\partial_x f^{n,m}(0)\|_\infty \equiv 0$, we use the representation for $\partial_x f^n$ from (4.5) and take the difference of this equation for $\partial_x f^n$ and $\partial_x f^m$ to find

$$
\begin{aligned}
|\partial_x f^{n,m}(t,x,v)| &= \left| \int_0^t (\partial_x B^{n-1} \partial_v f^n)(s, X^n(s), V^n(s))\, ds \right. \\
&\qquad \left. - \int_0^t (\partial_x B^{m-1} \partial_v f^m)(s, X^m(s), V^m(s))\, ds \right| \\
&\le \int_0^t \left| (\partial_x B^{n-1} \partial_v f^n)(s, X^n(s), V^n(s)) - (\partial_x B^{m-1} \partial_v f^n)(s, X^n(s), V^n(s)) \right. \\
&\qquad \left. + (\partial_x B^{m-1} \partial_v f^n)(s, X^n(s), V^n(s)) - (\partial_x B^{m-1} \partial_v f^m)(s, X^m(s), V^m(s)) \right|\, ds \\
&\le \int_0^t \left(\|\partial_x B^{n-1,m-1}(s)\|_\infty \|\partial_v f^n(s)\|_\infty + \|\partial_x B^{m-1}(s)\|_\infty \|\partial_v f^{n,m}(s)\|_\infty \right)\, ds.
\end{aligned}
$$

From Section 4.1 we have bounds on $\|\partial_v f^n(s)\|_\infty$ and $\|\partial_x B^{m-1}(s)\|_\infty$. So, taking the supremum

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in x and v and using (4.12) the above inequality becomes

$$\|\partial_x f^{n,m}(t)\|_\infty \le C \int_0^t \left[\|\partial_x B^{n-1,m-1}(s)\|_\infty + \int_0^s \|\partial_x f^{n,m}(\tau)\|_\infty \, d\tau \right] ds$$

which simplifies to

$$\|\partial_x f^{n,m}(t)\|_\infty \le C \left(\int_0^t \|\partial_x B^{n-1,m-1}(s)\|_\infty \, ds + t \int_0^t \|\partial_x f^{n,m}(\tau)\|_\infty \, d\tau \right).$$

Using Gronwall's Inequality, we find

(4.13)
$$\|\partial_x f^{n,m}(s)\|_\infty \le C \|\partial_x B^{n-1,m-1}(s)\|_\infty.$$

where C may depend upon the time of existence. Combining (4.11) and (4.13) yields

$$\|\partial_x B^{n,m}(t)\|_\infty \le C \int_0^t \|\partial_x B^{n-1,m-1}(s)\|_\infty \, ds$$

for any $n, m \in \mathbb{N}$. Now consider $m = n - 1$ in the previous equation so that we have

$$\|\partial_x B^{n,n-1}(t)\|_\infty \le C \int_0^t \|\partial_x B^{n-1,n-2}(s)\|_\infty \, ds$$

for any $n \in \mathbb{N}$ with $n \ge 2$. Using this recursive relation we deduce

$$\|\partial_x B^{n,n-1}(t)\|_\infty \le C \|\partial_x B^{1,0}(t)\|_\infty \frac{t^{n-1}}{(n-1)!}$$

for every $n \in \mathbb{N}$. Since $\|\partial_x B^1(t)\|_\infty$ and $\|\partial_x B^0(t)\|_\infty$ are bounded from Section 4.1, we have

$$\|\partial_x B^{n,m}(t)\|_\infty \le \sum_{k=m+1}^n \|\partial_x B^{k,k-1}(t)\|_\infty \le C \sum_{k=m+1}^n \frac{t^{k-1}}{(k-1)!} \to 0$$

as $n, m \to \infty$ for every $t > 0$. Therefore, $\partial_x B^n(t, x)$ is uniformly Cauchy for all $t > 0$ and $x \in \mathbb{R}$. Similarly, using (4.12) and (4.13) we see that $\|\partial_{(x,v)} f^{n,m}(t)\|_\infty \to 0$ as $n \to \infty$ for every $t > 0$

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and it follows that $\partial_{(x,v)} f^n(t, x, v)$ is uniformly Cauchy for all for every $t > 0, x, v \in \mathbb{R}$. The same argument can then be used to show that t-derivatives are also Cauchy. This property will be utilized in the next chapter as it implies that the sequences $\partial_{(t,x,v)} f^n$ and $\partial_{(t,x)} B^n$ must converge uniformly.

CONVERGENCE OF APPROXIMATIONS AND PROOF OF EXISTENCE

Now that the particle density and magnetic field along with their derivatives have been shown to be uniformly Cauchy, we may finish the proof of the theorem by collecting the work of Chapters 2 through 4 and proving the existence lemma:

Lemma 5.2 (Existence of Solution). *For any $f_0 \in C_c^1(\mathbb{R}^2)$, $B_0 \in C^1(\mathbb{R})$ there exist $T > 0$ and $f \in C^1([0,T] \times \mathbb{R}^2)$, $B \in C^1([0,T] \times \mathbb{R})$ such that*

(SVM)
$$\begin{cases} \partial_t f + v\partial_x f + B\partial_v f = 0, & x, v \in \mathbb{R}, t \in [0,T] \\[2mm] \partial_t B + \partial_x B = -\displaystyle\int f\,dv, & x \in \mathbb{R}, t \in [0,T] \end{cases}$$

and

(IC)
$$\begin{cases} f(0,x,v) = f_0(x,v), & x, v \in \mathbb{R} \\[2mm] B(0,x) = B_0(x), & x \in \mathbb{R}. \end{cases}$$

Proof of Existence: Let T_0 be the maximal existence time of the solution to (4.8). Using the Cauchy property of f^n, X^n, V^n, B^n, and their derivatives, we may conclude that each sequence of functions converges uniformly on the time interval $[0,T]$ for any $T < T_0$ and uniformly for $x, v \in \mathbb{R}$. Since the space of continuous functions is complete with respect to the norm of uniform convergence, we may conclude that these sequences converge to continuous functions. Therefore, let us define $f \in C([0,T] \times \mathbb{R}^2)$ by

$$f(t,x,v) = \lim_{n\to\infty} f^n(t,x,v) = \lim_{n\to\infty} f_0\big(X^n(0,t,x,v), V^n(0,t,x,v)\big).$$

Then, we similarly define the field

$$B(t,x) = \lim_{n\to\infty} B^n(t,x) = \lim_{n\to\infty}\left(B_0(x-t) + \int_0^t \int f^n(s, x-t+s, v)\,dv\,ds\right).$$

Thus, using the uniform convergence in t and v of f^n, we have for every $t \in [0, T]$, $x \in \mathbb{R}$

(5.1) $$B(t, x) = B_0(x - t) - \int_0^t \int f(s, x - t + s, v) \, dv \, ds.$$

Further, we define

$$X = \lim_{n \to \infty} X^n, \qquad V = \lim_{n \to \infty} V^n.$$

It follows from (3.1) and the uniform field bound that

(5.2) $$\begin{cases} \dfrac{\partial X}{\partial s} = V(s, t, x, v), \\ \dfrac{\partial V}{\partial s} = B(s, X(s, t, x, v)), \\ X(t, t, x, v) = x, \\ V(t, t, x, v) = v \end{cases}$$

and by the continuity of f_0, we see that

$$f(t, x, v) = f_0(X(0, t, x, v), V(0, t, x, v)),$$

whence for every $s \in [0, t]$,

(5.3) $$f(t, x, v) = f(s, X(s, t, x, v), V(s, t, x, v)).$$

Since the approximating sequence of derivatives (e.g., $\partial_x f^n$) of these functions converge uniformly, this implies that f and B are C^1 and their derivatives are necessarily the limits of the respective sequences, meaning

$$\partial_x f = \lim_{n \to \infty} \partial_x f^n, \quad \partial_v f = \lim_{n \to \infty} \partial_v f^n, \quad \partial_t f = \lim_{n \to \infty} \partial_t f^n,$$

$$\partial_x B = \lim_{n \to \infty} \partial_x B^n, \qquad \partial_t B = \lim_{n \to \infty} \partial_t B^n.$$

Using (5.1) and taking derivatives, we see that the field equation for B of (SVM) holds. Finally,

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taking derivatives of (5.3) and using (5.2), we see that the Vlasov equation of (SVM) holds. Additionally, the solutions (5.3) and (5.1) satisfy the initial conditions (IC). Therefore, the continuously differentiable functions f and B satisfy (SVM) with (IC) and there exists a solution to our original problem. $\qquad\square$

CHAPTER 6

CONCLUSION

In summary, we introduced the simplified Vlasov-Maxwell equations (SVM) which describe the physical phenomena of a collisionless plasma. Since little is known about the problem, specifically a justification of the validity of the model, we analyzed the equations from a mathematical perspective and answered two questions related to (SVM). First, we employed the method of successive approximations to prove the existence of solutions to this nonlinear system of partial differential equations on a local-time interval. After constructing bounds on the particle density and magnetic field, we proved the convergence of a sequence of linear approximations to the solutions of (SVM). Secondly, we showed that solutions to (SVM) are unique, a fairly important result as the existence of multiple solutions may suggest an ill-posed problem and refute the validity of these equations. A brief mathematical summary of our results are contained in Theorem 1.1, highlighting the precise statement of the existence and uniqueness of these solutions.

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BIOGRAPHICAL INFORMATION

Charles Nguyen is a senior majoring in mathematics at the University of Texas at Arlington (UTA) and will receive his B.A. degree in May 2010. Charlie is a member of the Mathematical Association of America (MAA), the Society for the Advancement of Chicanos and Native Americans in Science (SACNAS), the Texas Iota Chapter of Pi Mu Epsilon, and the Delta Rho Chapter of the Beta Theta Pi fraternity. He is currently a member of the Honors College and serves as the current president of the MAA Chapter at the University of Texas at Arlington. He has participated in the Rice University Summer Institute in Statistics (RUSIS) during the summer of 2009 and the Center for Undergraduate Research in Mathematics (CURM) undergraduate program during the $2009 - 2010$ academic year. He has given oral and poster presentations at various conferences such as the 2009 Texas Undergraduate Mathematics Conference at Sam Houston State University in Huntsville, TX, the 2009 SACNAS National Conference in Dallas, TX, the 2010 CURM Spring Research Conference at Brigham Young University in Provo, UT, and the Honors Undergraduate Research and Creative Activity Conference at UTA. Additionally, he will be presenting much of the work contained in this thesis at the 2010 MAA MathFest in Pittsburgh, PA. He has performed research under Prof. Pankavich's supervision on plasma dynamics, nonlinear partial differential equations, and the existence and uniqueness of solutions throughout the $2009 - 2010$ academic year and has been involved in collaborative research on these topics with two other undergraduate students, Robert Allen and Dustin Brewer, under Prof. Pankavich's mentorship and direction. He will pursue graduate studies in mathematics and statistics at the University of Texas at Arlington beginning in the Fall 2010 semester.