Retrieving the Green's function of the diffusion equation from the response to a random forcing

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It is known that the Green's function for nondissipative acoustic or elastic wave propagation can be extracted by correlating noise recorded at different receivers. This property is often related to the invariance for time reversal of the acoustic or elastic wave equations. The diffusion equation is not invariant for time reversal. It is shown in this work that the Green's function of the diffusion equation can also be retrieved by correlating solutions of the diffusion equation that are excited randomly and are recorded at different locations. This property can be used to retrieve the Green's function for diffusive systems from ambient fluctuations. Potential applications include the fluid pressure in porous media, electromagnetic fields in conducting media, the diffusive transport of contaminants, and the intensity of multiply scattered waves.

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I. INTRODUCTION

The Green's function for acoustic or elastic waves can be extracted by cross correlating recorded waves that are excited by a random excitation; see Ref. [1], for a tutorial. Derivations of this principle have been presented based on normal modes [2], on representation theorems [3–5], on time-reversal invariance [6,7], and on the principle of stationary phase [8–10]. This technique has found applications in ultrasound [11–13], crustal seismology [14–18], exploration seismology [19,20], structural engineering [21,22], and numerical modeling [23]. Recently, the extraction of the Green's function by cross correlation has been derived for general coupled systems of linear equations [24].

The principle of extracting the Green's function of a system from ambient fluctuations creates the possibility to retrieve the impulse response of a system without using controlled point sources. This impulse response can be used for imaging, tomography, or other methods to determine the properties of the medium. For example, models of the crust in California have been constructed using surface wave tomography based on microseismic noise [15,18]. The autocorrelation of ambient seismic noise has been used for daily monitoring of fault zones [25] and volcanoes [26]. The Green's function extracted from ambient noise can also be used to model the response of a system to a prescribed excitation without knowing the *in situ* properties of the system.

The extraction of the Green's function from ambient noise has been described extensively for wave propagation of acoustic or elastic waves without intrinsic attenuation (e.g., [1–10]). In the absence of intrinsic attenuation, the wave equation is invariant for time reversal, and several derivations of the reconstruction of the Green's function are indeed based on time-reversal invariance [6,7,20].

Many physical systems are not invariant under time reversal. Intrinsic attenuation breaks the symmetry for time reversal for acoustic and elastic wave propagation. Electrical conductivity breaks the time-reversal symmetry of Maxwell's equations. It has been shown theoretically [27] and observationally [21,22] that the impulse response of attenuating waves can be retrieved from ambient fluctuations.

Time-reversal invariance is, however, not essential for retrieving the Green's function from ambient noise. This can be seen by considering the diffusion equation

$$\frac{\partial p(\mathbf{r},t)}{\partial t} = \boldsymbol{\nabla}\cdot(D(\mathbf{r})\,\boldsymbol{\nabla}\,p(\mathbf{r},t)) + q(\mathbf{r},t), \tag{1}$$

where the diffusion constant D may depend on position $\mathbf{r}$. The diffusion equation is not invariant for time reversal because the operation $t\rightarrow -t$ changes the sign of the first term. Equation (1) is of practical importance because it describes conductive heat transport, diffusive transport of tracers and contaminants, fluid flow in porous media [28], electromagnetic waves in conducting media [29], and the energy transport of multiply scattered waves, e.g., [30].

The derivation in this work is applicable to the frequency domain, and the following Fourier convention is used: $p(\mathbf{r},t)=\int p(\mathbf{r},\omega)\exp(-i\omega t)d\omega$. With this convention, the diffusion equation is, in the frequency domain, given by

$$i\omega p(\mathbf{r},\omega) + \boldsymbol{\nabla}\cdot(D(\mathbf{r})\,\boldsymbol{\nabla}\,p(\mathbf{r},\omega)) = -q(\mathbf{r},\omega). \tag{2}$$

Time reversal corresponds, in the frequency domain, to complex conjugation. The time-reversed diffusion equation is thus given by

$$-i\omega p^*(\mathbf{r},\omega) + \boldsymbol{\nabla}\cdot(D(\mathbf{r})\,\boldsymbol{\nabla}\,p^*(\mathbf{r},\omega)) = -q^*(\mathbf{r},\omega), \tag{3}$$

where the asterisk denotes complex conjugation. The sign difference in the first terms of expressions (2) and (3) is due to the lack of time-reversal invariance of the diffusion equation.

It is shown here that the Green's function for the diffusion equation can be retrieved by correlating noise recorded at several locations in a diffusive system. One application of this technique is monitoring flow in porous media. We therefore refer to the solution of the diffusion equation as *pressure*, but the results are equally valid for other diffusive systems. In the following, all expressions are given in the frequency domain, and the frequency dependence is not shown explicitly.

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II. REPRESENTATION THEOREMS OF THE CONVOLUTION AND CORRELATION TYPE

Following Fokkema and van den Berg [31,32], we consider representation theorems of the convolution and correlation types by using expressions (2) and (3) for two solutions p_A and p_B with source terms q_A and q_B, respectively. The representation theorem of the convolution type is obtained by computing $\int(p_B E_A - p_A E_B)dV$, where E_A denotes Eq. (2) for state A, and where $\int(\cdots)dV$ denotes an integration over volume V. This gives

$$\int [p_B \nabla \cdot (D \nabla p_A) - p_A \nabla \cdot (D \nabla p_B)]dV = \int (p_A q_B - p_B q_A)dV. \quad (4)$$

Note that in the subtraction the $i\omega p$ terms cancel. Applying an integration by parts to the left-hand side of expression (4) and using Gauss's theorem gives

$$\oint D(p_B \nabla p_A - p_A \nabla p_B) \cdot d\mathbf{S} = \int (p_A q_B - p_B q_A)dV, \quad (5)$$

where the integral $\oint(\cdots)\cdot d\mathbf{S}$ is over the surface that bounds volume V.

A representation theorem of the correlation type can be obtained by evaluating $\int(p_B^* E_A - p_A Q_B)dV$, where Q_B denotes Eq. (3) for state B. Carrying out an integration by parts gives

$$\oint D(p_B^* \nabla p_A - p_A \nabla p_B^*) \cdot d\mathbf{S} + 2i\omega \int p_A p_B^* dV = \int (p_A q_B^* - p_B^* q_A)dV. \quad (6)$$

Note that now the $i\omega$ terms in expressions (2) and (3) do not cancel, but combine to give the volume integral in the left-hand side. The presence of this term results from the lack of invariance of time-reversal of the diffusion equation.

In the following integration over all space is used. The contribution of the surface integral vanishes because the solution of the diffusion equation $p(\mathbf{r},\omega)$ vanishes exponentially as $r \to \infty$. Therefore, the representation theorems (5) and (6) reduce to

$$\int (p_A q_B - p_B q_A)dV = 0 \quad (7)$$

and

$$\int (p_A q_B^* - p_B^* q_A)dV = 2i\omega \int p_A p_B^* dV. \quad (8)$$

The contribution of the surface integral also vanishes for a finite volume in case the solution satisfies either Dirichlet conditions ($p=0$), Neumann conditions ($\partial p/\partial n = 0$), or mixed boundary conditions ($\partial p/\partial n + ap = 0$) at the surface that bounds the volume.

III. REPRESENTATION THEOREMS AND GREEN'S FUNCTIONS

The Green's function for the diffusion equation is the solution to Eq. (2) when the forcing is a δ function,

$$i\omega G(\mathbf{r},\mathbf{r}_0) + \nabla \cdot (D(\mathbf{r}) \nabla G(\mathbf{r},\mathbf{r}_0)) = -\delta(\mathbf{r}-\mathbf{r}_0). \quad (9)$$

Setting $q_A(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_0)$ in expression (7) implies that $p_A(\mathbf{r}) = G(\mathbf{r},\mathbf{r}_0)$. For this choice of q_A, expression (7) reduces to

$$p(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}_0) q(\mathbf{r}_0) dV_0, \quad (10)$$

where the subscripts B are dropped. Alternatively, setting

$$q_{A,B}(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_{A,B}) \quad (11)$$

implies that

$$p_{A,B}(\mathbf{r}) = G(\mathbf{r},\mathbf{r}_{A,B}). \quad (12)$$

For this choice of the states A and B, expression (7) reduces to the reciprocity relation,

$$G(\mathbf{r}_B,\mathbf{r}_A) = G(\mathbf{r}_A,\mathbf{r}_B). \quad (13)$$

Inserting the states (12) into expression (8) gives

$$G(\mathbf{r}_B,\mathbf{r}_A) - G^*(\mathbf{r}_A,\mathbf{r}_B) = 2i\omega \int G(\mathbf{r},\mathbf{r}_A) G^*(\mathbf{r},\mathbf{r}_B)dV. \quad (14)$$

Using reciprocity, this expression can also be written as

$$G(\mathbf{r}_A,\mathbf{r}_B,\omega) - G^*(\mathbf{r}_A,\mathbf{r}_B,\omega) = 2i\omega \int G(\mathbf{r}_A,\mathbf{r},\omega) G^*(\mathbf{r}_B,\mathbf{r},\omega)dV. \quad (15)$$

The left-hand side of Eq. (15) corresponds, in the time domain, to the superposition of the Green's function and the time-reversed Green's function. In Sec. IV, we consider how this superposition can be retrieved from the cross correlation of the pressure generated by uncorrelated sources.

IV. RETRIEVING THE GREEN'S FUNCTION

In order to show how the Green's function can be extracted from the correlation of solutions generated by random sources, let us consider spatially uncorrelated sources with power spectrum $|q(\omega)|^2$ that does not depend on location,

$$\langle q(\mathbf{r}_1,\omega) q^*(\mathbf{r}_2,\omega)\rangle = \delta(\mathbf{r}_1-\mathbf{r}_2)|q(\omega)|^2, \quad (16)$$

where the angular brackets denote the expectation value. In practical applications, this expectation value is usually replaced by using several nonoverlapping time windows (e.g., [33,26]). Multiplying Eq. (15) with $|q(\omega)|^2$, the volume integral in that expression can be written as

$$
\begin{aligned}
&|q(\omega)|^2 \int G(\mathbf{r}_A,\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dV \\
&\quad = \int G(\mathbf{r}_A,\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)|q(\omega)|^2 G^*(\mathbf{r}_B,\mathbf{r}_2)dV_1dV_2 \\
&\quad = \int G(\mathbf{r}_A,\mathbf{r}_1)\langle q(\mathbf{r}_1,\omega)q^*(\mathbf{r}_2,\omega)\rangle G^*(\mathbf{r}_B,\mathbf{r}_2)dV_1dV_2 \\
&\quad = \langle (\int G(\mathbf{r}_A,\mathbf{r}_1)q(\mathbf{r}_1,\omega)dV_1)(\int G(\mathbf{r}_B,\mathbf{r}_2)q(\mathbf{r}_2,\omega)dV_2)^*\rangle .
\end{aligned}
\tag{17}
$$

When we use this result and expression (10), Eq. (15) after multiplication with $|q(\omega)|^2$ is given by

$$
(G(\mathbf{r}_A,\mathbf{r}_B,\omega)-G^*(\mathbf{r}_A,\mathbf{r}_B,\omega))|q(\omega)|^2 = 2i\omega\langle p(\mathbf{r}_A,\omega)p^*(\mathbf{r}_B,\omega)\rangle, \tag{18}
$$

where $p(\mathbf{r},\omega)$ is the pressure at location $\mathbf{r}$ due to the random forcing $q(\mathbf{r},\omega)$.

Equation (18) states that the superposition of the Green's function $G(\mathbf{r}_A,\mathbf{r}_B,\omega)$ and its time-reversed version is, after multiplication with the power spectrum of the excitation, equal to the correlation of the random fields at locations $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively. The prefactor $2i\omega$ corresponds, in the time domain, with $-2d/dt$. Since multiplication in the frequency domain corresponds, in the time domain, to convolution expression (19), in the time domain, is given by

$$
\begin{aligned}
&(G(\mathbf{r}_B,\mathbf{r}_A,t)-G(\mathbf{r}_B,\mathbf{r}_A,-t)) * C_q(t) \\
&\quad = -2\frac{d}{dt}\langle p(\mathbf{r}_A,t)\otimes p(\mathbf{r}_B,t)\rangle
\end{aligned}
\tag{19}
$$

where the asterisk denotes convolution, $\otimes$ denotes correlation, and $C_q(t)$ is the autocorrelation of $q(t)$.

V. DISCUSSION

The Green's function of the diffusion equation can be retrieved by cross correlating measurements of a diffusive system that is excited by random noise. Since the diffusion equation is not invariant for time reversal, this shows that invariance for time reversal is not essential for the retrieval of the Green's function by cross correlation.

For elastic and acoustic waves, the Green's function can be extracted from waves that are excited randomly at the surface that surrounds the volume [3,5]. This is not the case for the diffusion equation. For a volume of radius R, the surface area grows with R^2, but for a homogeneous medium the solution of the diffusion equation varies with the radius as $R^{-1}\exp(-\sqrt{\omega/2D}R)$. The contribution of the surface integral therefore depends on the radius of the volume, and the derivation shown here holds for an infinite volume ($R\rightarrow\infty$), or for a finite volume when Dirichlet, Neumann, or mixed boundary conditions hold at the surface that bounds the volume. In contrast to the retrieval of the Green's function for nonattenuating acoustic or elastic waves, where one needs random sources on a surface that bounds the volume, one needs random sources throughout the volume for the retrieval of the Green's function for the diffusion equation.

The theory presented here provides an example that time-reversal invariance is not required for the extraction of the Green's function from ambient fluctuations. The diffusion equation governs physical systems of practical importance, and the derivation presented here makes it possible to retrieve the impulse response of diffusive systems from measured fluctuations.

In this work, the phrase *pressure* is used for the solution of the diffusion equation because the pore pressure in a porous medium follows the diffusion equation. The theory of this work makes it possible to retrieve the Green's function for fluid flow in an aquifer or hydrocarbon reservoir from recorded pressure fluctuations. This Green's function can be used to estimate parameters such as hydraulic conductivity and to model the fluid transport in the subsurface without explicit knowledge of the *in situ* hydraulic conductivity. Similarly, the impulse response for the diffusive transport of contaminants can be retrieved from observations of ambient fluctuations in the concentration. The Green's function thus obtained can then be used to predict the diffusive transport of a localized release of the contaminant.

Electromagnetic fields in a conducting media satisfy the diffusion equation. This has been used in the magnetotelluric method where the ambient fluctuations in the electric and magnetic fields observed at one location are used to determine the electrical conductivity [34]. The theory presented here makes it possible to retrieve the Green's function for electromagnetic fields for noncoincident points from observed electromagnetic fluctuations.

The intensity of multiply scattered waves satisfies, for late times, the diffusion equation. Controlled intensity fluctuations of multiply scattered waves have been used to create images of the spatial distribution of the diffusion constant. This has found application in medical imaging, e.g., [35]. Instead of using controlled, spatially localized, sources for the intensity of scattered waves, one may use the theory of this work to use random spatially distributed sources instead.

As always, the application of the theory to these, and other, applications faces implementation issues. The assumption that the sources of the ambient fluctuations have a homogeneous spatial distribution may not be satisfied in practical applications. For applications where this condition is satisfied, the theory can be used to extract the impulse response of diffusive systems without using a controlled, localized, source.

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