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The reciprocity properties of geometrical spreading

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SUMMARY

Reciprocity is an important property of acoustic and elastic waves. In this work it is explicity verified that acoustic waves also satisfy the reciprocity theorem in a ray-geometric approximation. This is achieved by deriving a reciprocity relation for the geometric spreading. The analysis is based on integrating the equations of dynamic ray tracing from the source to a receiver and in the reverse direction. It is shown that for a point source the geometric spreading for rays travelling in opposite directions differs by a factor depending on the velocities at the endpoints of the ray. This factor depends on the number of dimensions that one considers. Since the equations of kinematic and dynamic ray tracing are the same for elastic waves and acoustic waves, the derived reciprocity relations for the geometrical spreading hold for elastic waves as well. The results obtained are used to correct some errors in the derivation of an averaging theorem by Snieder & Lomax (1996).

Key words: geometrical spreading, ray theory, reciprocity, wave propagation.

1 INTRODUCTION

It is well known that solutions of a variety of wave equations such as the Helmholtz equation, the acoustic wave equation or the elastic wave equation satisfy reciprocity. This implies that when a point source and a receiver are interchanged the recorded wavefield is the same. In practical applications, ray theory is an extremely powerful tool in solving forward and inverse problems in wave propagation. The question addressed in this work is whether the ray-geometric approximations to the solutions of the wave equations satisfy reciprocity as well. Using the symplectic properties of the equations of dynamic ray tracing it has been shown by Kendall, Guest & Thomson (1992) and Chapman & Coates (1994) that the ray-geometric approximation to the elastic wave equation satisfies reciprocity. Richards (1971) gives the reciprocity relation for geometrical spreading based on a proof of G. E. Backus that employs the reciprocity of traveltime and geometric considerations.

In this work the reciprocity properties of geometrical spreading are derived from the evolution equations for the wave-front curvature. It turns out that the reciprocity properties depend on the number of dimensions. The result is used to show that the ray-geometric approximation of the acoustic wave equation satisfies reciprocity. The analysis is based on the equations of dynamic ray tracing presented by Červený & Hron (1980), hereafter referred to as CH. It turns out that the geometrical spreading for rays travelling in opposite directions is different when the velocities at the endpoints of the ray are different. The velocity terms that enter when source and receiver are interchanged cancel other velocity terms in such a way that the ray-geometric solutions indeed satisfy reciprocity. The equations of dynamic ray tracing depend on the velocity only; the reciprocity relations derived here for the geometrical spreading for the acoustic wave equation therefore also hold for the elastic wave equation and the Helmholtz equation.

The acoustic wave equation and the dependence of the amplitude of the pressure field and the displacement field on velocity and density are introduced in Section 2. The ray-geometric Green's functions in two and three dimensions are derived in Section 3. The effect of interchanging a point source and a receiver on the geometrical spreading in two dimensions is derived in Section 4, while the corresponding result for three dimensions is presented in Section 5. In Section 6 it is shown that the ray-geometric Green's functions indeed satisfy reciprocity. The results derived in Sections 3–5 are used in Appendix A to correct some errors that occurred in the derivation of the averaging theorem by Snieder & Lomax (1996).

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2 THE DEPENDENCE OF THE AMPLITUDE ON DENSITY AND VELOCITY

In this section the acoustic wave equation is analysed:

$$\nabla\cdot\left(\frac{1}{\rho}\nabla p\right)+\frac{\omega^2}{\kappa}p=0\,, \tag{1}$$

where ρ is the density and κ the bulk modulus. The principles stated here can immediately be generalized to waves in an isotropic elastic medium. By setting the density constant, eq. (1) reduces to the Helmholtz equation. In a ray-geometrical treatment the pressure field is written as

$$p(\mathbf{r},\omega)=A(\mathbf{r},\omega)\exp[i\psi(\mathbf{r},\omega)]\,, \tag{2}$$

where the amplitude A and the phase ψ are real numbers. The standard ray-geometrical treatment proceeds by inserting (2) into (1) and by taking the terms of highest power in ω of the real part of the resulting equation; this gives the eikonal equation

$$|\nabla\psi|^2=\frac{\omega^2}{v^2}\,, \tag{3}$$

with the velocity v given by

$$v=\sqrt{\frac{\kappa}{\rho}}\,. \tag{4}$$

Similarly, by inserting (2) into (1) and taking the imaginary part one finds without making any approximation the transport equation:

$$\nabla\cdot\left(\frac{A^2\nabla\psi}{\rho}\right)=0\,. \tag{5}$$

From the eikonal eq. (3) it follows that

$$\nabla\psi=\frac{\omega}{v}\hat{\mathbf{n}}\,; \tag{6}$$

in this expression $\hat{\mathbf{n}}$ is the unit vector along the ray. By inserting this result in (5) it follows that the amplitude satisfies

$$A=C\frac{\sqrt{\rho v}}{\sqrt{J}}\,. \tag{7}$$

In this expression J is the geometrical spreading and C is a constant that depends on the source of the wavefield.

The important point of this expression is that the amplitude varies in proportion to $\sqrt{\rho v}$. However, the amplitude in expression (7) is for the pressure field. The amplitude of the displacement field follows by inserting the solution (2) and eq. (6) in Newton's law $\rho\omega^2\mathbf{u}=\nabla p$, and by retaining the terms of highest power in ω; this gives

$$\mathbf{u}=\frac{i\hat{\mathbf{n}}A}{\rho v\omega}\,e^{i\psi}\,. \tag{8}$$

This expression states that acoustic waves have a longitudinal polarization; they oscillate in the ray direction $\hat{\mathbf{n}}$. Using (7) it follows that the amplitude A_u of the displacement satisfies

$$A_u\sim\frac{1}{\sqrt{\rho v}\sqrt{J}}\,. \tag{9}$$

Comparing this result with the amplitude of the pressure field one sees that the displacement field and the pressure field differ by a factor ρv. This quantity is equal to the acoustic impedance; this result reflects the fact that the acoustic impedance is defined as the ratio between the pressure and the displacement. Note that the displacement amplitude in expression (9) has the same form as the displacement amplitude derived by CH for elastic waves.

3 RAY-GEOMETRIC GREEN'S FUNCTIONS IN TWO AND THREE DIMENSIONS

The Green's functions used here are solutions of

$$\nabla\cdot\left(\frac{1}{\rho}\nabla G(\mathbf{r},\mathbf{r}')\right)+\frac{\omega^2}{\kappa}G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')\,. \tag{10}$$

Using Green's theorem one readily shows that the Green's function satisfies the following reciprocity theorem:

$$G(\mathbf{r}_1,\mathbf{r}_2)=G(\mathbf{r}_2,\mathbf{r}_1)\,. \tag{11}$$

Sometimes the Green's function is not defined by eq. (10), but by a similar expression with the right-hand side multiplied by $1/v^2(\mathbf{r})$. The Green's functions defined in that way do not satisfy (11), but do satisfy a similar equation that contains additional terms $1/v^2(\mathbf{r}_1)$ and $1/v^2(\mathbf{r}_2)$.

It follows from (2) and (7) that in the ray-geometric limit the Green's function is given by

$$G(\mathbf{r}_1,\mathbf{r}_2)=C\frac{\sqrt{\rho(\mathbf{r}_1)v(\mathbf{r}_1)}}{\sqrt{J(\mathbf{r}_1,\mathbf{r}_2)}}\exp[i\omega\tau(\mathbf{r}_1,\mathbf{r}_2)]\,, \tag{12}$$

where the coefficient C is not yet determined. In this expression $\tau(\mathbf{r}_1,\mathbf{r}_2)$ is the traveltime of a wave that travels from $\mathbf{r}_2$ to $\mathbf{r}_1$, while $J(\mathbf{r}_1,\mathbf{r}_2)$ is the geometrical spreading at $\mathbf{r}_1$ for a point source at $\mathbf{r}_2$. The coefficient C follows by analysing eq. (12) close to the source. In doing so one can replace the medium by a homogeneous medium with the properties of the medium at the source. This part of the analysis depends on the number of dimensions. In the analysis one should account for the fact that (10) is not equivalent to the scalar wave equation. In order to establish the connection with the solutions of the scalar wave equation for a homogeneous medium, eliminate κ from (10) using (4) to rewrite (10) in the following form:

$$\rho\nabla\cdot\left(\frac{1}{\rho}\nabla G(\mathbf{r},\mathbf{r}')\right)+\frac{\omega^2}{v^2}G(\mathbf{r},\mathbf{r}')=\rho\delta(\mathbf{r}-\mathbf{r}')\,. \tag{13}$$

Note the density term on the right-hand side.

In three dimensions, the Green's function (13) in a homogeneous medium with the properties of the source at $\mathbf{r}_2$ is given by

$$G_{3D}^{\mathrm{hom}}(\mathbf{r}_1,\mathbf{r}_2)=-\frac{\rho(\mathbf{r}_2)}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\exp(i\omega|\mathbf{r}_1-\mathbf{r}_2|/v(\mathbf{r}_2))\,. \tag{14}$$

The density term arises from the right-hand side of (13). Close to the source the wavefield expands spherically and the geometrical spreading is given by $J(\mathbf{r}_1,\mathbf{r}_2)=|\mathbf{r}_1-\mathbf{r}_2|^2$ (as $\mathbf{r}_1\to\mathbf{r}_2$). Using this one finds comparing (12) and (14) that $C=-\sqrt{\rho(\mathbf{r}_2)}/(4\pi\sqrt{v(\mathbf{r}_2)})$. The ray-geometric Green's function is thus given by

$$G(\mathbf{r}_1,\mathbf{r}_2)=-\frac{1}{4\pi}\frac{\sqrt{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)v(\mathbf{r}_1)}}{\sqrt{v(\mathbf{r}_2)}}\frac{\exp[i\omega\tau(\mathbf{r}_1,\mathbf{r}_2)]}{\sqrt{J(\mathbf{r}_1,\mathbf{r}_2)}}\quad\text{(three dimensions).} \tag{15}$$

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Note that this Green's function can only satisfy reciprocity (11) when the geometric spreading has reciprocity properties that depend on the velocity at the endpoints of the ray.

In two dimensions the Green's function of a homogenous space with the properties of the source at $\mathbf{r}_2$ is given by Morse & Feshbach (1953):

$$G_{2\mathrm{D}}^{\mathrm{hom}}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{i}{4}\,\rho(\mathbf{r}_2) H_0^{(1)}\left(\frac{\omega}{v(\mathbf{r}_2)}\,|\mathbf{r}_1 - \mathbf{r}_2|\right), \tag{16}$$

with $H_0^{(1)}$ the zeroth-order Hankel function of the first kind. Again, the density term arises from the right-hand side of (13). Using the asymptotic expansion of the Hankel function ($H_0^{(1)}(x) \approx \exp i(x - \pi/4)/\sqrt{\pi x/2}$), and using the fact that in a 2-D medium the geometrical spreading is given by $J(\mathbf{r}_1, \mathbf{r}_2) = |\mathbf{r}_1 - \mathbf{r}_2|$ (as $\mathbf{r}_1 \to \mathbf{r}_2$), one finds by comparing (12) with (16) that $C = -\exp(i\pi/4)\sqrt{\rho(\mathbf{r}_2)}/\sqrt{8\pi\omega}$. The ray-geometric Green's function in two dimensions is thus given by

$$G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{e^{i\pi/4}}{\sqrt{8\pi\omega}}\sqrt{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)v(\mathbf{r}_1)}\,\frac{\exp[i\omega\tau(\mathbf{r}_1, \mathbf{r}_2)]}{\sqrt{J(\mathbf{r}_1, \mathbf{r}_2)}}$$

$$\text{(two dimensions)}. \tag{17}$$

Note that in contrast with the ray-geometric Green's function (14) for three dimensions, the 2-D ray-geometric Green's function depends only on the velocity $\mathbf{r}_1$ at the observation point but not on the velocity $v(\mathbf{r}_2)$ at the source point. This implies that reciprocity of the ray-geometric Green's functions can only be satisfied when the geometric spreading has different properties in two dimensions than in three dimensions when the source and the receiver are interchanged. For this reason the 2-D case and the 3-D case are analysed separately in the next two sections.

4 THE EFFECT OF CHANGING SOURCE AND RECEIVER ON THE GEOMETRICAL SPREADING IN TWO DIMENSIONS

The geometric spreading can be determined using the equations of dynamic ray tracing as given by CH. The geometric spreading satisfies

$$\frac{dJ}{ds} = KJ. \tag{18}$$

In this expression the scalar K denotes the wave-front curvature. As shown in CH this quantity satisfies a Ricatti equation:

$$\frac{dK}{ds} = \frac{1}{v}\frac{dv}{ds}K - K^2 - \frac{1}{v}v_{qq}, \tag{19}$$

where v_{qq} denotes the second derivative of the velocity perpendicular to the ray. Given the wave-front curvature, expression (18) can be integrated to give

$$\ln J(s) = \int^{s} K(s')ds'. \tag{20}$$

Let us now consider two rays: one ray runs from point A to point B [the quantities associated with this ray are given the superscript $(+)$], and the second ray runs in the reverse direction from point B to point A [the quantities associated with this ray are given the superscript $(-)$]; see Fig. 1. The arc length along the ray to point A is denoted by s, the total arc length of the ray is denoted by S. All quantities are measured with s as independent parameter, and wherever the arc length

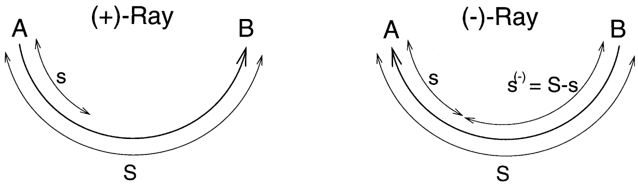

Figure 1. Definition of the geometric variables for rays travelling from A to B (left panel) and reverse rays running from B to A (right panel).

to point B occurs it is replaced by $s^{(-)} = S - s$. Eq. (20) for the geometrical spreading can be integrated when the initial conditions are specified. For a point source the initial conditions for the forward and the reverse ray are given by

$$J^{(+)}(s) = s \quad \text{as} \quad s \downarrow 0, \tag{21}$$

$$J^{(-)}(s) = S - s \quad \text{as} \quad s \uparrow S, \tag{22}$$

where the identity $s^{(-)} = S - s$ was used in the last expression.

Consider the wave-front curvature of the two rays. For the ray running from A to B the wave-front curvature satisfies (19):

$$\frac{dK^{(+)}}{ds} = \frac{1}{v}\frac{dv}{ds}K^{(+)} - K^{(+)2} - \frac{1}{v}v_{qq}. \tag{23}$$

The reverse ray satisfies the same equation, but with s replaced by $s^{(-)} = S - s$; using this to eliminate $s^{(-)}$ in favour of s, using $\partial/\partial s^{(-)} = -\partial/\partial s$, gives

$$\frac{dK^{(-)}}{ds} = \frac{1}{v}\frac{dv}{ds}K^{(-)} + K^{(-)2} + \frac{1}{v}v_{qq}. \tag{24}$$

Adding (23) and (24) and dividing by $(K^{(+)} + K^{(-)})$ gives

$$\frac{1}{K^{(+)} + K^{(-)}}\frac{d}{ds}(K^{(+)} + K^{(-)}) = \frac{1}{v}\frac{dv}{ds} - (K^{(+)} - K^{(-)}). \tag{25}$$

This expression can be integrated to give

$$\left[\ln\left(\frac{K^{(+)} + K^{(-)}}{v}\right)\right]_{s=0}^{s=S} = \int_0^S K^{(-)}ds - \int_0^S K^{(+)}ds. \tag{26}$$

The quantity on the left-hand side needs to be evaluated at the endpoints of the ray. From (18), (21) and (22) one finds that

$$K^{(+)}(s) = \frac{1}{s} \quad \text{as} \quad s \downarrow 0, \tag{27}$$

$$K^{(-)}(s) = \frac{1}{S - s} \quad \text{as} \quad s \uparrow S. \tag{28}$$

This means that at the point A ($s \downarrow 0$) the wave-front curvature $K^{(+)}$ dominates on the left-hand side of (26), so that regardless of the value of $K^{(-)}$,

$$\left[\ln\left(\frac{K^{(+)} + K^{(-)}}{v}\right)\right]_{s\downarrow 0} = \ln\left(\frac{1/s}{v(0)}\right) = \ln\left(\frac{1}{s}\right)_{s\downarrow 0} - \ln v(s=0); \tag{29}$$

similarly, at point B ($s \uparrow S$),

$$\left[\ln\left(\frac{K^{(+)} + K^{(-)}}{v}\right)\right]_{s\uparrow S} = \ln\left(\frac{1}{S - s}\right)_{s\uparrow S} - \ln v(s=S) = \ln\left(\frac{1}{s}\right)_{s\downarrow 0}$$

$$- \ln v(s=S): \tag{30}$$

Combining these results gives

$$\left[\ln\left(\frac{K^{(+)}+K^{(-)}}{v}\right)\right]_{s=0}^{s=S}=-\ln\left(\frac{v(s=S)}{v(s=0)}\right). \tag{31}$$

With (26) this gives

$$\int_0^S K^{(-)}ds-\int_0^S K^{(+)}ds=-\ln\left(\frac{v(s=S)}{v(s=0)}\right). \tag{32}$$

Expression (20) can be used to relate this result to the geometric spreading:

$$\ln J^{(-)}(s=0)-\ln J^{(+)}(s=S)=-\ln\left(\frac{v(s=S)}{v(s=0)}\right), \tag{33}$$

which can also be written as

$$\frac{J^{(-)}(s=0)}{v(s=0)}=\frac{J^{(+)}(s=S)}{v(s=S)}. \tag{34}$$

In order to change to a more general notation, let $J(\mathbf{r}_1, \mathbf{r}_2)$ be the geometrical spreading at point $\mathbf{r}_1$ for a point source at $\mathbf{r}_2$. Taking $\mathbf{r}_2$ to be the point A (i.e. $s=0$) and $\mathbf{r}_1$ to be the point B, expression (34) can be written as

$$\frac{J(\mathbf{r}_2, \mathbf{r}_1)}{v(\mathbf{r}_2)}=\frac{J(\mathbf{r}_1, \mathbf{r}_2)}{v(\mathbf{r}_1)} \quad \text{(point source in two dimensions)}. \tag{35}$$

It thus follows that the geometrical spreading is not reciprocal and that scale factors related to the velocity at the endpoints of the ray are needed to relate the geometrical spreading for rays travelling in opposite directions.

5 THE EFFECT OF CHANGING SOURCE AND RECEIVER ON THE GEOMETRICAL SPREADING IN THREE DIMENSIONS

For three dimensions the analysis is similar to the derivation of the previous section. The only difference is that the wave-front curvature is now characterized by a 2×2 wave-front curvature matrix $\mathbf{K}$ rather than the scalar K, see CH for details. Instead of (18) the geometric spreading in three dimensions satisfies

$$\frac{dJ}{ds}=\mathrm{tr}\mathbf{K}\, J, \tag{36}$$

where $\mathrm{tr}\mathbf{K}$ denotes the trace of $\mathbf{K}$. This expression can be integrated to give

$$\ln J(s)=\int^s \mathrm{tr}\mathbf{K}\, ds'. \tag{37}$$

As shown in CH, the wave-front curvature matrix satisfies a (matrix) Ricatti equation:

$$\frac{d\mathbf{K}}{ds}=\frac{1}{v}\frac{dv}{ds}\mathbf{K}-\mathbf{K}^2-\frac{1}{v}\mathbf{V}, \tag{38}$$

where $\mathbf{V}$ is the 2×2 matrix of second derivatives of the velocity perpendicular to the reference ray, $V_{ij}\equiv\partial^2 v/\partial q_i\partial q_j$, with q_i the ray-centred coordinates.

As in the previous section the superscript $(+)$ denotes quantities for a ray running from point A to point B, while the superscript $(-)$ refers to quantities associated with the ray in the reverse direction. Analogously to expressions (23) and (24) the curvature matrices for the forward and reverse ray satisfy

$$\frac{d\mathbf{K}^{(+)}}{ds}=\frac{1}{v}\frac{dv}{ds}\mathbf{K}^{(+)}-\mathbf{K}^{(+)2}-\frac{1}{v}\mathbf{V}, \tag{39}$$

$$\frac{d\mathbf{K}^{(-)}}{ds}=-\frac{1}{v}\frac{dv}{ds}\mathbf{K}^{(-)}+\mathbf{K}^{(-)2}+\frac{1}{v}\mathbf{V}. \tag{40}$$

By direct substitution of these expressions one can verify that

$$\frac{d}{ds}\det(\mathbf{K}^{(+)}+\mathbf{K}^{(-)})=\frac{2}{v}\frac{dv}{ds}\det(\mathbf{K}^{(+)}+\mathbf{K}^{(-)})+(\mathrm{tr}\mathbf{K}^{(+)}-\mathrm{tr}\mathbf{K}^{(-)})\det(\mathbf{K}^{(+)}+\mathbf{K}^{(-)}). \tag{41}$$

This expression can be integrated to give

$$\left[\ln\left(\frac{\det(\mathbf{K}^{(+)}+\mathbf{K}^{(-)})}{v^2}\right)\right]_{s=0}^{s=S}=\int_0^S \mathrm{tr}\mathbf{K}^{(-)}ds-\int_0^S \mathrm{tr}\mathbf{K}^{(+)}ds. \tag{42}$$

As $s\downarrow 0$, $\mathbf{K}^{(+)}$ dominates the contribution of $\mathbf{K}^{(-)}$, and just as in the previous section the value of the left-hand side of the $\mathbf{K}^{(+)}$ contribution as $s\downarrow 0$ is cancelled by the $\mathbf{K}^{(-)}$ contribution for $s\uparrow S$ so that

$$\int_0^S \mathrm{tr}\mathbf{K}^{(-)}ds-\int_0^S \mathrm{tr}\mathbf{K}^{(+)}ds=-\ln\left(\frac{v^2(s=S)}{v^2(s=0)}\right). \tag{43}$$

With (37) this gives

$$\frac{J^{(-)}(s=0)}{v^2(s=0)}=\frac{J^{(+)}(s=S)}{v^2(s=S)}. \tag{44}$$

Reverting now to a more general notation where $\mathbf{r}_2$ is associated with the point A ($s=0$) and $\mathbf{r}_1$ with the point B ($s=S$) this result can be written as

$$\frac{J(\mathbf{r}_2, \mathbf{r}_1)}{v^2(\mathbf{r}_2)}=\frac{J(\mathbf{r}_1, \mathbf{r}_2)}{v^2(\mathbf{r}_1)} \quad \text{(point source in three dimensions)}. \tag{45}$$

This result is equivalent to the reciprocity relation (18) given by Richards (1971) that was derived from the reciprocity of traveltimes and geometric considerations. Comparing (45) with the corresponding results (35) for two dimensions one sees that the geometrical spreading has different reciprocity properties in different dimensions.

6 RECIPROCITY OF THE RAY-GEOMETRIC GREEN'S FUNCTIONS

Using the relations (35) and (45) one can show that the ray-geometric Green's functions (15) and (17) indeed satisfy the reciprocity relation (11). For two dimensions, interchanging $\mathbf{r}_1$ and $\mathbf{r}_2$ in the Green's function (17) gives

$$G(\mathbf{r}_2, \mathbf{r}_1)=-\frac{e^{i\pi/4}}{\sqrt{8\pi\omega}}\sqrt{\rho(\mathbf{r}_2)\rho(\mathbf{r}_1)v(\mathbf{r}_2)}\,\frac{\exp[i\omega\tau(\mathbf{r}_2, \mathbf{r}_1)]}{\sqrt{J(\mathbf{r}_2, \mathbf{r}_1)}}. \tag{46}$$

The traveltime is the integral of the slowness along the ray; this integral does not depend on the direction of integration, so that

$$\tau(\mathbf{r}_2, \mathbf{r}_1)=\tau(\mathbf{r}_1, \mathbf{r}_2). \tag{47}$$

Using this result and (35) for the reciprocity of the geometrical spreading one obtains, with (17),

$$G(\mathbf{r}_2, \mathbf{r}_1) = -\frac{e^{i\pi/4}}{\sqrt{8\pi\omega}} \sqrt{\rho(\mathbf{r}_2)\rho(\mathbf{r}_1)v(\mathbf{r}_2)} \sqrt{\frac{v(\mathbf{r}_1)}{v(\mathbf{r}_2)}} \frac{\exp[i\omega\tau(\mathbf{r}_1, \mathbf{r}_2)]}{\sqrt{J(\mathbf{r}_1,\mathbf{r}_2)}}$$

$$= G(\mathbf{r}_1, \mathbf{r}_2) . \quad (48)$$

For three dimensions the Green's function (15) with source and receiver interchanged is given by

$$G(\mathbf{r}_2, \mathbf{r}_1) = -\frac{1}{4\pi} \frac{\sqrt{\rho(\mathbf{r}_2)\rho(\mathbf{r}_1)v(\mathbf{r}_2)}}{\sqrt{v(\mathbf{r}_1)}} \frac{\exp[i\omega\tau(\mathbf{r}_2, \mathbf{r}_1)]}{\sqrt{J(\mathbf{r}_2, \mathbf{r}_1)}} . \quad (49)$$

Using (47) for the reciprocity of the traveltime and (45) for the reciprocity of the geometrical spreading one finds, with (15), that

$$G(\mathbf{r}_2, \mathbf{r}_1) = -\frac{1}{4\pi} \frac{\sqrt{\rho(\mathbf{r}_2)\rho(\mathbf{r}_1)v(\mathbf{r}_2)}}{\sqrt{v(\mathbf{r}_1)}} \frac{v(\mathbf{r}_1)}{v(\mathbf{r}_2)} \frac{\exp[i\omega\tau(\mathbf{r}_1, \mathbf{r}_2)]}{\sqrt{J(\mathbf{r}_1, \mathbf{r}_2)}}$$

$$= G(\mathbf{r}_1, \mathbf{r}_2) . \quad (50)$$

This implies that the ray-geometric Green's functions in two and three dimensions indeed satisfy reciprocity. Note that in establishing these results it was crucial that the velocity enters in the reciprocity relations (35) and (45) for the geometrical spreading in two and three dimensions respectively.

7 PHYSICAL INTERPRETATION OF THE VELOCITY TERMS IN THE RECIPROCITY RELATION OF THE GEOMETRICAL SPREADING

The velocity terms that appear in the reciprocity relations (35) and (45) may appear to be unnatural. However, a simple example will show why the velocity enters these relations. Consider a point source in a medium where the velocity depends on depth only and assume that the velocity increases with depth. For simplicity we consider a ray that travels vertically downwards or upwards; see Fig. 2. For the downward-travelling ray in the left panel of Fig. 2, the rays diverge more rapidly than they would for a homogeneous medium because rays curve away from high velocities. This means that the geometrical spreading for this ray is larger than it would be for a homogenous medium. Conversely, for the upward-travelling ray on the right-hand side of Fig. 2 the paraxial rays curve towards the central ray, again because the rays curve away from high velocities. This means that for the upward-travelling ray the geometrical spreading is less that it would be for a homogeneous medium. It is for this reason that the geometrical spreading cannot satisfy a reciprocity relation such as $J(\mathbf{r}_1, \mathbf{r}_2) = J(\mathbf{r}_2, \mathbf{r}_1)$ without velocity-dependent factors.

The example presented here also makes it possible to understand why the reciprocity relations for geometrical spreading in two and three dimensions are different. In three dimensions, the rays shown in Fig. 2 diverge (or converge) in two spatial directions, whereas in two dimensions the rays diverge (or converge) in only one spatial direction. For this reason, the geometrical spreading in three directions is the square of the geometrical spreading in two dimensions. Indeed, the velocity terms in the relation (45) for three dimensions are the square of the velocity factors in (35) for two dimensions.

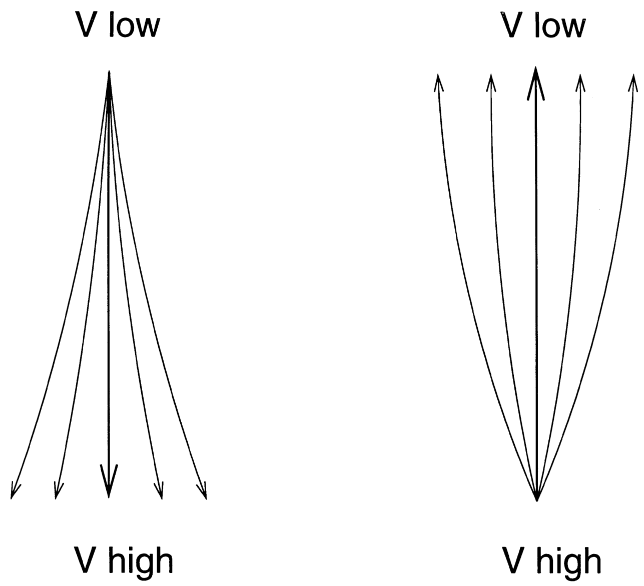

Figure 2. Paraxial rays for a reference ray running vertically downwards (left panel) and a ray running vertically upwards (right panel) for a medium where the laterally homogeneous velocity increases with depth.

The reciprocity relations (35) and (45) for the geometrical spreading only hold for a point source. The reason is that in the steps going from (26) to (29) the wave-front curvature $K^{(+)}$ dominates the wave-front curvature $K^{(-)}$ at the source point A so that the value of the wave-front curvature $K^{(-)}$ is irrelevant. However, this is only the case for a point source which gives a singular wave-front curvature $K^{(+)}$ at the source. For a plane wave, the wave-front curvature at the source would be $K^{(+)} = 0$, and the value of the wave-front curvature $K^{(-)}$ of the reverse ray would be important. The example shown in Fig. 2 allows us to understand why the reciprocity relations (35) and (45) cannot hold for a plane wave. Suppose one replaces the point source in Fig. 2 by a plane-wave source. A plane wave travelling upwards or downwards remains a plane wave, so that both the upward- and the downward-travelling plane waves satisfy $J(\mathbf{r}_1, \mathbf{r}_2) = J(\mathbf{r}_2, \mathbf{r}_1) = 1$, rather than (35) or (45) for two or three dimensions respectively.

8 CONCLUSIONS

The reciprocity relations (35) and (45) imply that the geometrical spreading is not invariant when source and receiver are interchanged. This reflects the physical fact that when a wave is focused by the variations in the velocity travelling one way, it is defocused when it travels in the opposite direction. It is shown here that the change in the geometrical spreading when source and receiver are interchanged depends only on the velocity at the endpoints of the ray. Curiously, this change in the geometrical spreading is balanced by other velocity-dependent terms in the response, so that reciprocity of the wavefield is valid within geometrical optics.

One should note that for elastic waves the equations of kinematic ray tracing and dynamic ray tracing are identical to the equations used here. This means that the geometrical spreading satisfies the same equations as analysed in Sections 4 and 5. This implies that the reciprocity relations (35) and (45)

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for two and three dimensions, respectively, are valid for elastic waves as well.

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REFERENCES

Červený, V. & Hron, F., 1980. The ray series method and dynamic ray tracing system for three-dimensional inhomogeneous media, *Bull. seism. Soc. Am.,* **70,** 47–77.

Chapman, C.H. & Coates, R.T., 1994. Generalized Born scattering in anisotropic media, *Wave Motion,* **19,** 309–341.

Kendall, J.M., Guest, W.S. & Thomson, C.J., 1992. Ray theory Green's function reciprocity and ray-centred coordinates in anisotropic media, *Geophys. J. Int.,* **108,** 364–371.

Morse, P. & Feshbach, H., 1953. *Methods of Theoretical Physics,* Part 1, McGraw-Hill, New York, NY.

Richards, P.G., 1971. An elasticity theorem for heterogeneous media, with an example of body wave dispersion in the Earth, *Geophys. J. R. astr. Soc.,* **22,** 453–472.

Snieder, R. & Lomax, A., 1996. Wavefield smoothing and the effect of rough velocity variations on arrival times and amplitudes, *Geophys. J. Int.,* **125,** 796–812.

APPENDIX A: CORRECTIONS TO THE WORK OF SNIEDER & LOMAX

Snieder & Lomax (1996) (hereafter referred to as SL) showed that for media with velocity perturbations that are so rough that the requirements for the validity of ray theory are violated, the phase of the wavefield is to first order given by a weighted average of the velocity perturbation over the first Fresnel zone with a weight function that follows from theory. Although their derivation is correct for the case of a homogeneous reference medium, their work contains some errors for the case of an inhomogeneous reference medium. These errors are corrected in this appendix. The equations in the work of SL are referred to with the prefix 'SL'. SL studied the Helmholtz equation:

$$\nabla^2 u + \frac{\omega^2}{v^2(\mathbf{r})}(1+n(\mathbf{r}))u = 0 . \tag{A1}$$

This expression is equivalent to (1) when one assumes when the density is set to a constant value. The relative perturbation in $1/v^2$ is denoted by the perturbation $n(\mathbf{r})$. In the following, the 3-D case is analysed first.

SL incorrectly assumed in (SL6.1) that the amplitude varies in proportion to $1/\sqrt{v}$, whereas it is shown in (7) that the amplitude is proportional to $\sqrt{v}$. This means that expression (SL6.6) should be replaced by the ray-geometric Green's function (15). In addition, SL assumed that the geometric spreading contained in the Green's function satisfied $J(\mathbf{r}_0, \mathbf{r}) = J(\mathbf{r}, \mathbf{r}_0)$, whereas it is shown in (45) that this is not correct. Using the fact that the Green's function satisfies reciprocity ($G(\mathbf{r}_0, \mathbf{r}) = G(\mathbf{r}, \mathbf{r}_0)$) and using the correct Green's function (15) one arrives at an expression identical to eq. (SL6.8). This implies that SL obtained the right result (SL6.8) for the wrong reason. The remainder of Section 6 of SL is correct.

However, in the steps leading to (SL7.1) one should use the relation

$$\frac{A(s)}{A(s_0)} = \sqrt{\frac{J(s_0)v(s)}{J(s)v(s_0)}} , \tag{A2}$$

rather than (SL6.1). [This expression follows immediately from (7)]. Taking this into account, eq. (SL7.1) for the 3-D case should be corrected to read

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2v(\mathbf{r}_0)} u_0(\mathbf{r}_0) \times \int \frac{\displaystyle\iint \frac{\sqrt{J(s_0)}}{\sqrt{J(s)}\sqrt{J(s,s_0)}} n(\mathbf{r}) \frac{\exp[i\omega T(\mathbf{r},\mathbf{r}_0)]}{\sqrt{\det(\mathbf{K}^{\rm in}+\mathbf{K}^{\rm out})}} h(s,q_1,q_2) dq_1 dq_2}{\displaystyle\iint \exp[i\omega T(\mathbf{r},\mathbf{r}_0)] dq_1 dq_2} ds . \tag{A3}$$

This implies that the factor $v(s_0)/v(s)$ in (SL7.1) should be deleted. Using this in the subsequent analysis of SL, one finds that (SL7.3) should be replaced by

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2} u_0(\mathbf{r}_0) \int \frac{\displaystyle\iint \frac{n(\mathbf{r})}{v(\mathbf{r})} \exp[i\omega T(\mathbf{r},\mathbf{r}_0)] h(s,q_1,q_2) dq_1 dq_2}{\displaystyle\iint \exp[i\omega T(\mathbf{r},\mathbf{r}_0)] dq_1 dq_2} ds . \tag{A4}$$

Compared to (SL7.3) one finds that the term $v^2(s_0)/v^2(s)$ is absent and that $n(\mathbf{r})$ is divided by $v(\mathbf{r})$ rather than $v(\mathbf{r}_0)$.

For the 2-D case treated in SL similar corrections need to be made. The Green's function (SL8.1) should be replaced by (17) and reciprocity should be applied to the Green's function $G(\mathbf{r}_0, \mathbf{r})$ rather than to $J(\mathbf{r}_0, \mathbf{r})$. Taking this into account one finds that (SL 8.3) must be replaced by

$$u_B(\mathbf{r}_0) = \omega\sqrt{\frac{\omega}{8\pi}} e^{i\pi/4} u_0(\mathbf{r}_0) \times \int \left\{ \int \frac{A(\mathbf{r})}{A(\mathbf{r}_0)} \frac{1}{v^{3/2}(\mathbf{r})} n(\mathbf{r}) \frac{\exp[i\omega T(\mathbf{r},\mathbf{r}_0)]}{\sqrt{J(\mathbf{r},\mathbf{r}_0)}} h(s,q) dq \right\} ds . \tag{A5}$$

The difference with (SL8.3) is that the terms $v(\mathbf{r})\sqrt{v(\mathbf{r}_0)}$, in the denominator of (SL8.3) are replaced by $v^{3/2}(\mathbf{r})$. Taking this into account in the subsequent analysis, one finds that (SL8.5) must be replaced by

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2} u_0(\mathbf{r}_0) \times \int \frac{\displaystyle\int \frac{A(\mathbf{r})}{A(\mathbf{r}_0)} \frac{\sqrt{v(s)}}{v^{3/2}(\mathbf{r})} n(\mathbf{r}) \frac{\exp[i\omega T(\mathbf{r},\mathbf{r}_0)]}{\sqrt{J(\mathbf{r},\mathbf{r}_0)}} \frac{h(s,q)}{\sqrt{K^{\rm in}+K^{\rm out}}} dq}{\displaystyle\int \exp[i\omega T(\mathbf{r},\mathbf{r}_0)] dq} ds . \tag{A6}$$

Hence, the velocity-dependent terms should be $\sqrt{v(s)}/v^{3/2}(\mathbf{r})$ instead of $\sqrt{v(\mathbf{r}_0)v(s)}/v(\mathbf{r})v(\mathbf{r}_0)$. The final result (SL8.8) for the

2-D case should therefore be replaced by

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2}\, u_0(\mathbf{r}_0) \int \frac{\displaystyle\int \frac{n(\mathbf{r})}{v(s)} \exp[i\omega T(\mathbf{r}, \mathbf{r}_0)] h(s, q) dq}{\displaystyle\int \exp[i\omega T(\mathbf{r}, \mathbf{r}_0)] dq}\, ds\,. \quad \text{(A7)}$$

Note that the expressions (54) and (57) for three and two dimensions, respectively, have the same form. In both expressions the perturbation enters the Born field (and hence the phase shift) through the combination $\omega n/2v$, despite the fact that in the intermediate steps of the derivation the velocity terms multiplying $n(\mathbf{r})$ are different in two and in three dimensions. The factor $\omega n/2v$ can be understood as follows. It follows from (51) that the local wavenumber is given by $k^2 = (\omega^2/v^2(\mathbf{r}))(1 + n(\mathbf{r}))$. The wavenumber can be written as the sum of a reference wavenumber $\omega/v(\mathbf{r})$ plus a perturbation δk: $k = \omega/v(\mathbf{r}) + \delta k$. Inserting this in the above expression one finds to first order in the perturbation $n(\mathbf{r})$:

$$\delta k = \frac{\omega}{v(\mathbf{r})} \{\sqrt{1 + n(\mathbf{r})} - 1\} = \frac{\omega n(\mathbf{r})}{2v(\mathbf{r})}\,. \quad \text{(A8)}$$

This implies that with the corrections to the theory of SL presented in this appendix the correct wavenumber perturbation is retrieved for both the 2-D case and the 3-D case.

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