

Lagrangian Green's function extraction, with applications to potential fields, diffusion and acoustic waves

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Abstract. The extraction of the response of physical systems from field fluctuations is an area undergoing rapid growth. It is of relevance because it makes it possible to obtain the response of the system from passive field fluctuations instead of from an active point source. The impulse response is characterized by the Green's function $G(t)$. The existing theory leads to the retrieval of $G(t) - G(-t)$, the difference of Green's function and its time reversed counterpart. We show that the theory for Green's function extraction can be extended to many problems to yield both the difference and the sum of $G(t)$ and $G(-t)$. The importance of this derivation is that it extends the type of sources of field fluctuations that can be used for Green's function retrieval. The new formalism also opens up the possibility to extract the Green's function from field fluctuations in static systems. We show how this can be done for potential field problems and for direct current problems in conducting media, and illustrate this with a numerical example. For diffusive fields, the new formalism provides the possibility to extract the Green's function from field fluctuations excited either by injection sources or by current sources; previously this could be done only with injection sources. We show that with the new theory, the Green's function of acoustic waves can be retrieved from field fluctuations excited by body

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forces instead of from injection sources when the attenuation is constant. The Green's function extraction formulated here corresponds, for acoustic waves, to a Lagrangian formulation rather than the Hamiltonian (energy) principles that were used previously.

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1. Introduction

Green's function extraction from field fluctuations is an area that has undergone spectacular growth recently [1]. The technique is ultimately based on the fluctuation dissipation theorem [2]–[5] formulated decades ago, but applications flourished when ultrasound measurements showed that this technique could be used in practice [6]–[9]. The technique found many applications that include ocean acoustics [10]–[12], crustal seismology [13]–[15], exploration seismology [16]–[20], structural engineering [21]–[25] and medical diagnostics [26]. There are several derivations for the theory of Green's function extraction that hold for a large class of linear scalar and vector systems [27]–[30].

Derivations for Green's function extraction yield either the sum or the difference of the causal Green's function $G(t)$ and its time-reversed counterpart $G(-t)$. In the frequency domain, this corresponds to retrieving $G(\omega) + G^*(\omega)$ or $G(\omega) - G^*(\omega)$, where the superscript asterisk denotes the complex conjugate. For example, for acoustic waves some derivations [31] lead to $G(\omega) + G^*(\omega)$, while others [32] give $G(\omega) - G^*(\omega)$. Part of this confusion is related to ambiguities in the definition of the Green's function. Suppose that one derivation for Green's function extraction gives an expression for $G(\omega) - G^*(\omega)$. Suppose that one defines a new Green's function $\tilde{G}(\omega)$ that is, in the frequency domain, related to the original Green's function $G(\omega)$ by the relation $\tilde{G}(\omega) = -i\omega G(\omega)$, where the factor $-i\omega$ accounts for the time derivative. The original expression for $G(\omega) - G^*(\omega)$ now leads to a new expression for $(-i\omega)^{-1}(\tilde{G}(\omega) + \tilde{G}^*(\omega))$. Similarly, when one arbitrarily includes a factor i in the excitation of the Green's function [31], one obtains a different superposition of $G(\omega)$ and $G^*(\omega)$ than

when such a factor i is excluded [32]. These differences in the definition of the Green's function depend purely on convention and have no physical meaning.

There is, however, a more fundamental reason for analyzing the extraction of the same Green's function $G(\omega)$ from either $G(\omega) + G^*(\omega)$ or $G(\omega) - G^*(\omega)$. For wave systems, one can extract the Green's function either from fluctuations excited by sources on a bounding surface [31], or from sources throughout the volume [8]. We show in this work that this difference is related to the extraction of different superpositions of $G(\omega)$ and $G^*(\omega)$. The different formulations for Green's function extraction are thus applicable to the extraction of the impulse response from field fluctuations excited by different types of sources. The new formalism presented here thus makes it possible to extract the Green's function for field fluctuations generated by random sources that were not included in previous formulations of Green's function extraction. One such application is potential fields. For these static problems the Green's function is real, so $G(\omega) - G^*(\omega) = 0$, and therefore the Green's function extraction *must* be based on the sum $G(\omega) + G^*(\omega)$. This new application was the original motivation for this work, and it theoretically opens up the possibility to obtain the impulse response of potential field problems from observed quasi-static field fluctuations.

In section 2, we analyze general scalar linear systems and generalize an earlier derivation for $G(\omega) - G^*(\omega)$ [28] to the extraction of the sum $G(\omega) + G^*(\omega)$. We apply this general formalism in section 3 to systems that are invariant under time reversal and show that for such systems $G(\omega) - G^*(\omega)$ can be obtained from sources of the field fluctuations on a surface that encloses the receivers, while $G(\omega) + G^*(\omega)$ follows from field fluctuations excited by volume sources. In section 4, we apply the general theory to electrostatics or direct current measurement in conducting media. We show that the static Green's function for such problems can be obtained from quasi-static field fluctuations excited by slowly varying electric dipoles throughout the volume. We illustrate the extraction of the electrostatic Green's function with a numerical example in section 5 that is based on the electric field of a dipole within a conducting shell as derived in appendix A. We apply the general theory to the diffusion equation in section 6. Field fluctuations for diffusive fields can be created either by injection sources that locally change the value of the field or by current sources that change the current of the field. We show that $G(\omega) - G^*(\omega)$ can be extracted from injection sources throughout the volume, while $G(\omega) + G^*(\omega)$ extends the previous theory by also allowing current sources in the volume. In sections 7.1 and 7.2, we consider acoustic waves and show that $G(\omega) - G^*(\omega)$ and $G(\omega) + G^*(\omega)$ can be extracted from field fluctuations, and discuss the physical meaning of these formulations. In appendix B, we relate this to the energy of the harmonic oscillator. We verify the expression for $G(\omega) + G^*(\omega)$ for acoustic waves in closed systems in appendix C using a normal mode expansion.

2. General theory

In this work, we consider fields that satisfy the following scalar equation:

$$\left(a_n(\mathbf{r}, t) * \frac{\partial^n}{\partial t^n} + \dots + a_1(\mathbf{r}, t) * \frac{\partial}{\partial t} \right) * u(\mathbf{r}, t) = H(\mathbf{r}, \nabla, t) * u(\mathbf{r}, t) + q(\mathbf{r}, t). \quad (1)$$

In this expression, the asterisk denotes temporal convolution, $H(\mathbf{r}, \nabla, t)$ is a spatial differential operator, and $q(\mathbf{r}, t)$ is the source of the field $u(\mathbf{r}, t)$. Table 1 gives examples of a_n and H for a variety of physical systems. Using the Fourier convention $f(t) = \int f(\omega) \exp(-i\omega t) d\omega$,

Table 1. Examples of a_n and H for different physical systems and the possibilities for Green's function retrieval. The symbols denote the following physical quantities: κ = compressibility, D = diffusion parameter, ρ = mass density, ε = electric permittivity and γ = inverse of quality factor.

Equation	a_n	H	$G - G^*$	$G + G^*$
Diffusion	$a_1 = 1$	$\nabla \cdot (D(\mathbf{r})\nabla)$	Injection sources in volume	Current sources in volume
Acoustic waves (no attenuation)	$a_2 = \kappa(\mathbf{r}, \omega)$ $(\text{Im}(\kappa) = 0)$	$\nabla \cdot (\rho^{-1}(\mathbf{r})\nabla)$	Injection sources on boundary	Difference of correlations
Acoustic waves (with attenuation)	$a_2 = \kappa(\mathbf{r}, \omega)$ $(\text{Im}(\kappa) \neq 0)$	$\nabla \cdot (\rho^{-1}(\mathbf{r})\nabla)$	Injection sources in volume	Volume forces ($\nabla\gamma = 0$)
Electrostatics	all $a_n = 0$	$\nabla \cdot (\varepsilon(\mathbf{r})\nabla)$	Not possible	Dipole sources
Systems invariant under time reversal	$\text{Re}(a_n) = 0$ for n odd $\text{Im}(a_n) = 0$ for n even	$H = H^*$	Sources on boundary	Sources in volume

expression (1) corresponds, in the frequency domain, to

$$\sum_n a_n(\mathbf{r}, \omega)(-i\omega)^n u(\mathbf{r}, \omega) = H(\mathbf{r}, \nabla, \omega)u(\mathbf{r}, \omega) + q(\mathbf{r}, \omega). \quad (2)$$

The derivation in the remainder of this work is in the frequency domain, and for brevity we suppress this frequency dependence in all equations. Functions without arguments depend on ω and \mathbf{r} .

Field equation (2) has many solutions that correspond to different values of the excitation $q(\mathbf{r}, \omega)$. We refer to these solutions as *field states*. In the following, we consider combinations of field states and distinguish these field states with subscripts A and B . An expression for $G - G^*$ follows by taking the field equation for a state A and multiplying it with the complex conjugate of a field state B . Interchanging the states A and B , taking the complex conjugate, subtracting the resulting expressions and using delta function sources for the two fields gives [28]

$$\begin{aligned} G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) &= -2 \sum_{n \text{ odd}} (-i\omega)^n \int_V \text{Re}(a_n(\mathbf{r}))G(\mathbf{r}, \mathbf{r}_A)G^*(\mathbf{r}, \mathbf{r}_B) dV \\ &\quad - 2i \sum_{n \text{ even}} (-i\omega)^n \int_V \text{Im}(a_n(\mathbf{r}))G(\mathbf{r}, \mathbf{r}_A)G^*(\mathbf{r}, \mathbf{r}_B) dV \\ &\quad + \int_V (G^*(\mathbf{r}, \mathbf{r}_B)H(\mathbf{r}, \nabla)G(\mathbf{r}, \mathbf{r}_A) - G(\mathbf{r}, \mathbf{r}_A)H^*(\mathbf{r}, \nabla)G^*(\mathbf{r}, \mathbf{r}_B)) dV, \end{aligned} \quad (3)$$

where Re and Im denote the real and imaginary parts, respectively. (For brevity we suppress the dependence of H on frequency in the above expression and the following.) Using a similar derivation, but using the addition of G and G^* instead of subtraction, one arrives at the following expression for $G + G^*$:

$$\begin{aligned} G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) &= 2i \sum_{n \text{ odd}} (-i\omega)^n \int_V \text{Im}(a_n(\mathbf{r}))G(\mathbf{r}, \mathbf{r}_A)G^*(\mathbf{r}, \mathbf{r}_B) dV \\ &\quad + 2 \sum_{n \text{ even}} (-i\omega)^n \int_V \text{Re}(a_n(\mathbf{r}))G(\mathbf{r}, \mathbf{r}_A)G^*(\mathbf{r}, \mathbf{r}_B) dV \\ &\quad - \int_V (G^*(\mathbf{r}, \mathbf{r}_B)H(\mathbf{r}, \nabla)G(\mathbf{r}, \mathbf{r}_A) + G(\mathbf{r}, \mathbf{r}_A)H^*(\mathbf{r}, \nabla)G^*(\mathbf{r}, \mathbf{r}_B)) dV, \end{aligned} \quad (4)$$

Note that H is, in general, a differential operator, hence the two terms in the last volume integral in this expression are different. Equation (3) and its application to the diffusion equation, acoustic waves and quantum mechanics was discussed earlier [28]. In the following, we discuss the connection between expressions (3) and (4) and show that equation (4) opens up new possibilities for Green's function extraction that are not covered by expression (3).

3. Systems that are invariant under time reversal

A connection has been established between time reversal and the Green's function extraction of undamped acoustic waves [33]. Because of the importance of time-reversal invariance to a large number of applications [34], we first analyze systems that are invariant under time reversal. Time reversal corresponds, in the frequency domain, to complex conjugation. For such systems equation (2) is equal to its complex conjugate; hence

$$H = H^*, \quad \text{Re}(a_n) = 0 \quad \text{for } n \text{ odd}, \quad \text{Im}(a_n) = 0 \quad \text{for } n \text{ even.} \quad (5)$$

Using this in expression (3) gives

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \int_V (G^*(\mathbf{r}, \mathbf{r}_B)H(\mathbf{r}, \nabla)G(\mathbf{r}, \mathbf{r}_A) - G(\mathbf{r}, \mathbf{r}_A)H(\mathbf{r}, \nabla)G^*(\mathbf{r}, \mathbf{r}_B)) dV. \quad (6)$$

Following [28], we define the operator L via

$$\int_V (fHg - gHf) dV = \oint_{\partial V} L(f, g) dS. \quad (7)$$

For example, consider

$$H(\mathbf{r}, \nabla)f(\mathbf{r}) = \nabla \cdot (B(\mathbf{r})\nabla f(\mathbf{r})); \quad (8)$$

as shown in table 1 this covers a large number of applications for different choices of $B(\mathbf{r})$. Using Gauss' theorem one finds, for this case, that

$$\int_V (fHg - gHf) dV = \oint_{\partial V} B(\mathbf{r}) \left(f \frac{\partial g}{\partial n} - \frac{\partial f}{\partial n} g \right) dS; \quad (9)$$

hence for this case $L(f, g) = B(\mathbf{r})(f \partial g / \partial n - \partial f / \partial n g)$. We are not aware of a proof that an operator L can be found for any operator H ; hence the existence of an operator L that satisfies equation (7) must be verified on a case-by-case basis. If the operator H is such that an operator can be found that satisfies expression (7), then it follows, for systems invariant under time reversal, that

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \oint_{\partial V} L(G^*(\mathbf{r}, \mathbf{r}_B), G(\mathbf{r}, \mathbf{r}_A)) dS. \quad (10)$$

Since the right-hand side contains a surface integral only, it is possible for systems that are invariant under time reversal and that satisfy expression (7) to extract the Green's function from field fluctuations that are excited by sources acting on a closed surface that surrounds \mathbf{r}_A and \mathbf{r}_B , e.g. [28, 31].

Let us next consider the combination $G + G^*$ for systems invariant under time reversal. Using the conditions (5) in expression (4) gives

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 2 \sum_n (-i\omega)^n \int_V a_n(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \\ - \int_V (G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}, \nabla) G(\mathbf{r}, \mathbf{r}_A) + G(\mathbf{r}, \mathbf{r}_A) H(\mathbf{r}, \nabla) G^*(\mathbf{r}, \mathbf{r}_B)) dV, \quad (11)$$

where, for odd n , we used the fact that $i \operatorname{Im}(a_n) = a_n$ (because $\operatorname{Re}(a_n) = 0$), whereas for even n the imaginary part of a_n vanishes, so $\operatorname{Re}(a_n) = a_n$. The first term in the right-hand side is symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$. Green's function extraction is based on the correlation of fields $u(\mathbf{r}_A)$ and $u(\mathbf{r}_B)$ that are excited by random sources. These fields are given by the Green's function (or its gradient), acting on these sources [1]. The resulting correlation is, in the frequency domain, given by $u(\mathbf{r}_A)u^*(\mathbf{r}_B)$. Since both fields depend in the same way on the Green's functions $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$, respectively, the correlation is symmetric in these Green's functions. By contrast, a term such as $G^*(\mathbf{r}, \mathbf{r}_B)H(\mathbf{r}, \nabla)G(\mathbf{r}, \mathbf{r}_A)$ is not symmetric in $G^*(\mathbf{r}, \mathbf{r}_B)$ and $G(\mathbf{r}, \mathbf{r}_A)$ because $H(\mathbf{r}, \nabla)$ is a differential operator that acts only on the field on its right. The field $G^*(\mathbf{r}, \mathbf{r}_B)$ is the field excited by a point source, while $H(\mathbf{r}, \nabla)G(\mathbf{r}, \mathbf{r}_A)$ corresponds, in general, to the field generated by a different type of source. The combination $G^*(\mathbf{r}, \mathbf{r}_B)H(\mathbf{r}, \nabla)G(\mathbf{r}, \mathbf{r}_A)$ therefore does not correspond to the correlation of field generated by the *same* type of source. The terms in the second volume integral can, however, sometimes be made symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$ using integration by parts. For example, for the operator of expression (8):

$$\int_V (fHg + gHf) dV = \oint_{\partial V} B \left(f \frac{\partial g}{\partial n} + \frac{\partial f}{\partial n} g \right) dS - 2 \int_V B(\nabla f \cdot \nabla g) dV. \quad (12)$$

As shown in table 1, this special case covers many physical systems for different choices of $B(\mathbf{r})$.

It follows from expression (12) that for the operator H from expression (8), the sum $G + G^*$ can be written in a way where each term in the right-hand side is symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$, but one cannot avoid a volume integral in the right-hand side. This means that Green's function extraction applied to $G + G^*$ from field fluctuations requires sources of these fluctuations throughout the volume. This leads to the important conclusion that for systems that are invariant under time reversal and with H given by equation (8), one can extract $G - G^*$ from field fluctuations excited on the surface ∂V , whereas for extraction of $G + G^*$ one needs field fluctuations excited by sources throughout the volume. The use of $G + G^*$ thus theoretically opens up the possibility of Green's function extraction of systems invariant for time reversal from field fluctuations excited by volume sources. Table 1 summarizes, for the examples shown in the work, the types of sources needed for the retrieval of $G - G^*$ or $G + G^*$.

4. Green's function extraction in electrostatics

In linear dielectric media, the electric displacement \mathbf{D} is related to the electric field \mathbf{E} by the relation $\mathbf{D} = \varepsilon \mathbf{E}$, with $\varepsilon(\mathbf{r})$ being the electrical permittivity [35]. Using the field equation $(\nabla \cdot \mathbf{D}) = q$, with $q(\mathbf{r})$ being the charge density, and the relation $\mathbf{E} = -\nabla u$, with $u(\mathbf{r})$ being the

electric potential, the following field equation is obtained:

$$0 = \nabla \cdot (\varepsilon(\mathbf{r}) \nabla u(\mathbf{r})) + q(\mathbf{r}). \quad (13)$$

In this section and the next, the field $u(\mathbf{r})$ is the electrostatic potential. In the notation of expression (2), all $a_n = 0$, while H is given by expression (8) with B replaced by ε . We make this replacement throughout this section. If instead of the electrostatic problem we consider direct currents in a conducting medium, then the charge density $q(\mathbf{r})$ is replaced by a volume density of charge injection or extraction rate $-\dot{q}(\mathbf{r})$ (the minus sign comes from the historical convention that the loss of charge from the source region constitutes a positive electric current), and the electric permittivity $\varepsilon(\mathbf{r})$ is replaced by the conductivity $\sigma(\mathbf{r})$. This leaves equation (13) intact with different symbols [36]. This results in a formulation for direct current methods in conducting media.

First note that the Green's function solution to expression (13) is real, so $G - G^* = 0$, and therefore one cannot retrieve G from the difference $G - G^*$. Expression (4) together with equation (12) leads, however, to the following nontrivial relation for $G + G^*$:

$$\begin{aligned} G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = & - \oint_{\partial V} \varepsilon(\mathbf{r}) \left(G(\mathbf{r}, \mathbf{r}_A) \frac{\partial G^*(\mathbf{r}, \mathbf{r}_B)}{\partial n} + \frac{\partial G(\mathbf{r}, \mathbf{r}_A)}{\partial n} G^*(\mathbf{r}, \mathbf{r}_B) \right) dS \\ & + 2 \int_V \varepsilon(\mathbf{r}) (\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV. \end{aligned} \quad (14)$$

We now assume either that the potential G or the normal component of the electrical field $E_n = -\partial G/\partial n$ vanishes at ∂V or that the boundary is at infinity. In those cases, the surface integral vanishes and

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 2 \int_V \varepsilon(\mathbf{r}) (\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV. \quad (15)$$

In order to establish the connection of this equation with the Green's function extraction from field fluctuations we use the field generated by an electric dipole distribution $\mathbf{p}(\mathbf{r})$ [35]

$$u(\mathbf{r}_0) = \int_V (\nabla G(\mathbf{r}_0, \mathbf{r})) \cdot \mathbf{p}(\mathbf{r}) dV. \quad (16)$$

We next consider random dipole sources that are spatially and directionally uncorrelated and satisfy

$$\langle p_i(\mathbf{r}_1) p_j(\mathbf{r}_2) \rangle = |S|^2 \varepsilon(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij}, \quad (17)$$

where $|S|^2$ measures the strength of the dipole sources. For the moment we consider an ensemble of identical electrostatic systems, each with their own excitation by dipoles, and $\langle \dots \rangle$ denotes the average over this ensemble of identical electrostatic systems that are excited by different dipoles. Multiplying expression (15) with $|S|^2$, using the fact that for this problem G is real and using the summation convention, we obtain

$$\begin{aligned} G(\mathbf{r}_A, \mathbf{r}_B) |S|^2 &= |S|^2 \int_V \varepsilon(\mathbf{r}) \partial_i G(\mathbf{r}, \mathbf{r}_A) \partial_i G^*(\mathbf{r}, \mathbf{r}_B) dV \\ &= \int_V \int_V |S|^2 \varepsilon(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij} \partial_i G(\mathbf{r}_1, \mathbf{r}_A) \partial_j G^*(\mathbf{r}_2, \mathbf{r}_B) dV_1 dV_2 \\ &= \left\langle \int_V \partial_i G(\mathbf{r}_1, \mathbf{r}_A) p_i(\mathbf{r}_1) dV_1 \int_V \partial_j G^*(\mathbf{r}_2, \mathbf{r}_A) p_j(\mathbf{r}_2) dV_2 \right\rangle \\ &= \langle u(\mathbf{r}_A) u^*(\mathbf{r}_B) \rangle, \end{aligned} \quad (18)$$

where the identity $\int f_i(\mathbf{r})g_j(\mathbf{r})dV = \iint f_i(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta_{ij}g_j(\mathbf{r}_2)dV_1dV_2$ has been used in the second identity, expression (17) in the third equality and expression (16) in the last identity. Expression (18) states that the electrostatic Green's function $G(\mathbf{r}_A, \mathbf{r}_B)$ follows from the ensemble average of the correlation of field fluctuations recorded at \mathbf{r}_A and \mathbf{r}_B that are excited by uncorrelated dipole sources.

In reality one may not have an ensemble of identical electrostatic systems, but one may have a system where random sources fluctuate with time. When the characteristic time of the temporal variations in these dipole sources is large compared with the time it takes for light to propagate through the system, the response of the system is quasi-static. In that case, the ensemble average can be replaced by a temporal average over the field fluctuations. In fact, the approach to replace an ensemble average by an average over time is common in seismology where averaging over multiple non-overlapping time windows is used to extract the dynamic Green's function [14, 37, 38]. By applying the same principle to quasi-static field fluctuations, one can extract the electrostatic Green's function from temporal field fluctuations.

5. Numerical simulation of Green's function retrieval in electrostatics

In this example, we illustrate the theory for the Green's function extraction for the electrostatic potential by cross-correlating the fields generated by random electric dipoles within a conducting spherical shell with radius R at which the potential vanishes. Using the method of images, we show in appendix A that the potential generated by a dipole \mathbf{p} at location \mathbf{r} inside the shell vanishes at the shell $r = R$ when one adds the fields generated by a monopole $q' = (R/r^2)(\mathbf{p} \cdot \hat{\mathbf{r}})$ and a dipole $\mathbf{p}' = -(R/r)^3(\mathbf{p} - 2(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})$, both at location $\mathbf{r}' = (R/r)^2\mathbf{r}$ outside the shell. We use a system of units scaled in such a way that $4\pi\epsilon_0$ and R are both equal to 1.

The Green's function extraction is applied to equation (18). In each realization, the field $u(\mathbf{r})$ is generated by ten random dipoles at locations that are drawn from a uniform distribution within the sphere. Each component from each dipole is drawn from a uniform distribution between -1 and $+1$. In equation (18), we use $\mathbf{r}_B = (0.3, 0.2, 0)$ and choose \mathbf{r}_A along the x -axis of a coordinate system that has its origin at the center of the sphere. Figure 1 shows the potential in 200 realizations at three points along the x -axis. The potential at each point has the character of white noise, which is not surprising because the dipoles in every realization are uncorrelated and have zero mean. The potential at the different locations is, however, correlated. It is these correlations in the fields generated by random sources that contain the information that ultimately leads to the extraction of the Green's function. Note that it does not matter whether one considers figure 1 to show different realizations or whether it shows a time series of the quasi-static electric response of a system that exhibits quasi-random electric dipoles as a function of time.

Figure 2 shows the cross-correlation of 50 000 realizations of the field fluctuations recorded at location $\mathbf{r}_B = (0.3, 0.2, 0)$ and various locations along the x -axis. In all examples shown, the realizations are divided in ten bins that each contain 5000 realizations that are drawn from the same statistical distribution. The cross-correlation of field fluctuations in the different bins is used to compute the mean of the cross-correlation of field fluctuations and the standard deviation in this mean [39]. The standard deviation of the mean is shown with the error bars in figure 2. The mean and error thus computed do not depend much on the number of bins chosen, which means that the averaging over the bins has the same statistical effect as the averaging over

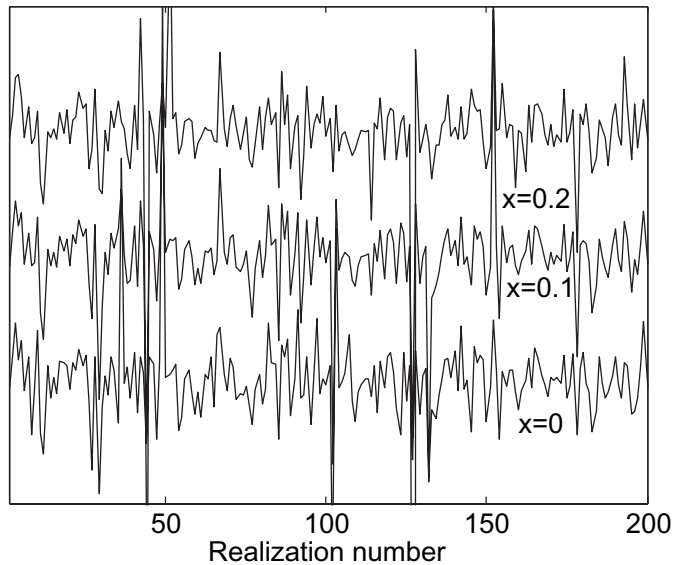


Figure 1. Two hundred realizations of the potential at the x -axis at $x = 0, 0.1,$ and $0.2,$ respectively.

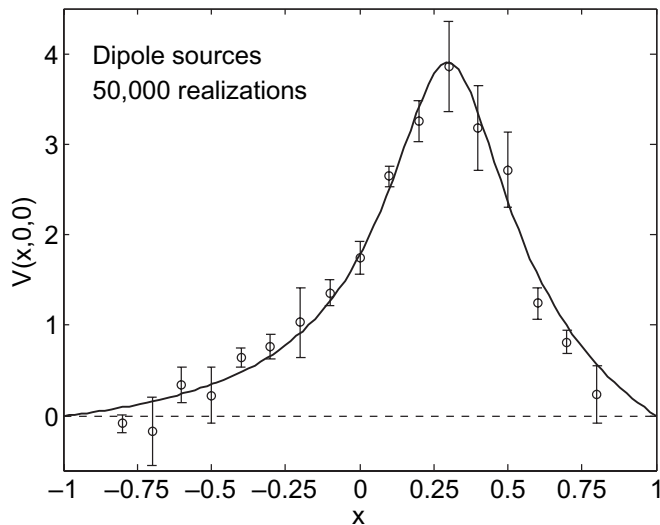


Figure 2. Open symbols: the potential and its standard deviation reconstructed from 50 000 realizations of random dipoles. Solid line: the true potential for a monopole at $(0.3, 0.2, 0)$.

realizations within each bin. The solid line in figure 2 shows the potential due to a unit point charge at location \mathbf{r}_B , which is the desired Green's function. The cross-correlations of field fluctuations, and their errors, are multiplied with a common scale factor that minimizes the difference of the cross-correlation and the Green's function. This scale factor accounts for the term $|S|^2$ in equation (18). Note that the potential estimated from the cross-correlation of field fluctuations generated by random dipole sources agrees with the true Green's function within

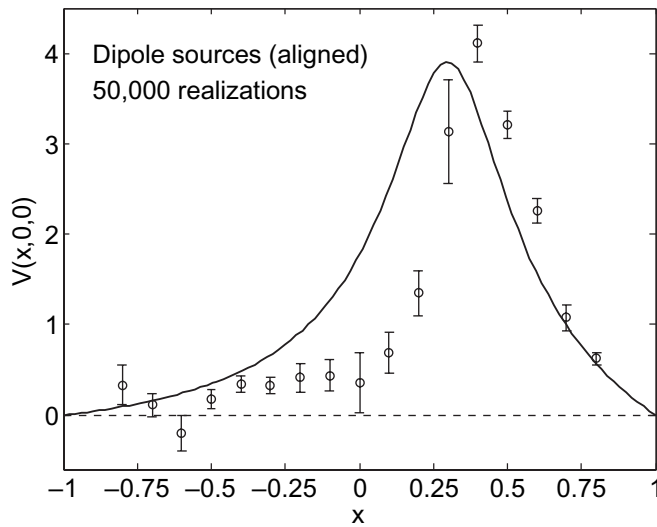


Figure 3. Open symbols: the potential and its standard deviation reconstructed from 50 000 realizations of aligned dipoles with dipole moment $\mathbf{p} = (1, 1, 1)$. Solid line: the true potential for a monopole at $(0.3, 0.2, 0)$.

the shown standard deviations. This confirms that the Green's function for this static example can indeed be extracted from the cross-correlation of field fluctuations.

The Green's function in the example of figure 2 was extracted from the cross-correlation of field fluctuations excited by random dipoles. In each realization, the field was generated by the simultaneous action of ten dipoles. In order to investigate what happens when the dipoles do not have a random orientation, we repeated the numerical experiment of figure 2, but now used the fixed dipole $\mathbf{p} = (1, 1, 1)$ in every realization. Ten of these dipoles were placed randomly within the spherical shell in every realization. The estimate of the Green's function from the cross-correlations of the associated field fluctuations is shown in figure 3. In this case, cross-correlation of field fluctuations does not lead to an acceptable estimate of the Green's function. The reason for this discrepancy is that, according to equation (17), the orientation of different dipole vectors must be uncorrelated. This assumption is violated when a constant dipole vector is used for every dipole.

One might think that it does not really matter whether field fluctuations are generated by dipoles or by monopoles (point charges). We show in figure 4 the field extracted from cross-correlation of 10 000 realizations of field fluctuations generated by random monopoles. In each realization, ten point charges are placed at uniformly chosen random positions within the shell. Each point charge is drawn from a uniform distribution between -1 and 1 . Note that the standard deviation in figure 4 is much smaller than that in figure 2, despite the fact that five times less realizations are used. There are two reasons for this. Firstly, for random dipole orientations, one carries out an implicit averaging over the direction of the dipole vectors, but such averaging is not needed for monopole sources. Secondly, the dipole fields vary more rapidly with space than the monopole field do, so the dipole fields must be sampled more finely by point charges to mimic the volume integrals in expression (18). Note that the cross-correlation of field fluctuations caused by random monopoles does not lead to an acceptable extraction of the Green's function.

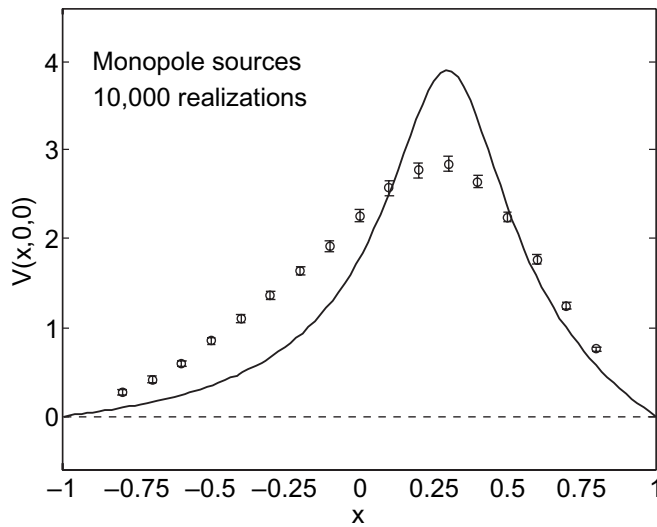


Figure 4. Open symbols: the potential and its standard deviation reconstructed from 10 000 realizations of random monopoles. Solid line: the true potential for a monopole at $(0.3, 0.2, 0)$.

The examples of figures 2 and 4 illustrate the paradox that one needs field fluctuations excited by random dipoles to extract the monopole field. This is fortunate; natural field fluctuations cannot be caused by monopoles because the random occurrence of monopoles is not consistent with charge conservation. Charges are neither created nor destroyed in a source-free region. This means that only dipoles or higher-order multipoles can excite field fluctuations. For the expected localized fluctuations, where the local charge separation is orders of magnitude smaller than that of the measurement scale, the contribution of the dipole moments dominates the potential field, and the electric potential is given by equation (16). The occurrence of dipole moments in a material can have four basic causes: electronic, ionic, dipolar and space-charge polarization [40]. Relaxation times of the first three processes are usually smaller than $1 \mu\text{s}$, whereas for the last process the relaxation time can be as large as 1 s. These timescales are extremely long compared with the propagation time of light through a system of the size of a laboratory experiment or geophysical field experiments. Field fluctuations can be generated, for example in natural rocks by electromagnetic radiation in fracturing rocks or in stressed rocks before fracturing. During fracturing, bonds are broken. The larger the number of cut bonds, the larger the number of excited atoms and hence the greater the electromagnetic radiation amplitude. These electromagnetic oscillations behave like surface vibrational optical waves, where positive charges move together in a diametrically opposite phase to the negative ones and decay exponentially into the material like Rayleigh waves. The resulting oscillating electric dipole is the source of the electromagnetic radiation. The pulse amplitude decays due to an interaction with bulk phonons, and the lifetime of the measurable electric field varies between several microseconds and $100 \mu\text{s}$ [41]. Random field fluctuations caused by charge separation have been observed in water-saturated porous media that were drained [42]. Each charge separation in that system is thought to be caused by the burst of a meniscus in the pore space, the so-called *Haines jump*. The application to direct current resistivity problems

and the connection with the fluctuation–dissipation theorem are discussed elsewhere in more detail [43].

6. Different types of noise sources and the diffusion equation

In this section, we compare different strategies for Green’s function extraction for the diffusion equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) + q, \quad (19)$$

with $D(\mathbf{r})$ being the diffusion parameter. In the notation of equation (1), $a_1 = 1$, all other a_n vanish, and H is given by expression (8) with $B(\mathbf{r})$ being the diffusion parameter $D(\mathbf{r})$. For simplicity, we consider the situation where either the field u or the current $D \partial u / \partial n$ vanishes on the boundary ∂V . This is, for example, the case when the boundary is at infinity.

Using expression (3) and equation (9) to convert the volume integral into a vanishing surface integral gives the following expression for $G - G^*$ [44]:

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\omega \int_V G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV. \quad (20)$$

Alternatively, equation (4) can be used to derive the following expression for $G + G^*$:

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 2 \int_V D(\mathbf{r}) (\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV, \quad (21)$$

where equation (12) has been used to convert a volume integral into a vanishing surface integral.

Both expressions can be used to extract the Green’s function from field fluctuations. As shown earlier [44], for uncorrelated sources that satisfy $\langle q(\mathbf{r}_1) q^*(\mathbf{r}_2) \rangle = \delta(\mathbf{r}_1 - \mathbf{r}_2) |S(\omega)|^2$, with $|S(\omega)|^2$ being the power spectrum of the sources, the difference $G - G^*$ can be retrieved from the correlation of the field fluctuations

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \frac{2i\omega}{|S(\omega)|^2} \langle u(\mathbf{r}_A) u^*(\mathbf{r}_B) \rangle, \quad (22)$$

where $u(\mathbf{r}_A) = \int G(\mathbf{r}_A, \mathbf{r}) q(\mathbf{r}) dV$ are the field fluctuations at \mathbf{r}_A generated by sources $q(\mathbf{r})$. A similar expression holds for $u(\mathbf{r}_B)$. Using reasoning similar to that used in deriving expression (18) one can extract $G + G^*$ by assuming that random spatially and directionally uncorrelated current sources \mathbf{J} are present that satisfy

$$\langle J_i(\mathbf{r}_1) J_j^*(\mathbf{r}_2) \rangle = |S(\omega)|^2 D(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij}. \quad (23)$$

The current sources generate field fluctuations $u^{(J)}$, and using the same arguments as used in the derivation of expression (18), it follows that

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = \frac{2}{|S(\omega)|^2} \langle u^{(J)}(\mathbf{r}_A) u^{(J)*}(\mathbf{r}_B) \rangle. \quad (24)$$

Expressions (22) and (24) show that both $G - G^*$ and $G + G^*$ can be extracted from field fluctuations, but for $G - G^*$ these fluctuations must be excited by sources of the field itself (monopole sources), whereas for $G + G^*$ the fluctuations must be excited by current sources (dipole sources). The new formalism for $G + G^*$ opens up the possibility to extract the Green’s function for diffusive systems from field fluctuations excited by current sources.

7. Acoustic waves

As the next example we treat acoustic waves where the pressure p and sources q are related by

$$\kappa(\mathbf{r}, t) * \frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = \nabla \cdot \left(\frac{1}{\rho(\mathbf{r})} \nabla p(\mathbf{r}, t) \right) + q(\mathbf{r}, t), \quad (25)$$

with ρ being the mass density and κ the compressibility. For attenuating media the compressibility is a relaxation function. This corresponds, in the frequency domain, to a frequency-dependent compressibility that, because of the Kramer–Kronig relation, has a nonzero imaginary part. In the notation of expression (1), $a_2 = \kappa$, all other a_n vanish, and H is given by equation (8) with B replaced by $1/\rho$ (see table 1). In the following subsections, we analyze the pressure generated by injection rate sources and body forces, so we first discuss the pressure response generated by both types of sources. The excitation q is related to the injection rate sources Q and body forces \mathbf{f} by the relation $q = \dot{Q} - \nabla \cdot (\mathbf{f}/\rho)$, where the dot denotes the time derivative. With the employed Fourier convention, this corresponds, in the frequency domain, to

$$q(\mathbf{r}) = -i\omega Q(\mathbf{r}) - \nabla \cdot \left(\frac{\mathbf{f}(\mathbf{r})}{\rho(\mathbf{r})} \right). \quad (26)$$

For a general excitation $q(\mathbf{r})$, the pressure is given by $p(\mathbf{r}_0) = \int G(\mathbf{r}_0, \mathbf{r}) q(\mathbf{r}) dV$. Using the decomposition (26) in injection sources and body forces, the pressure can be written as

$$p(\mathbf{r}_0) = -i\omega \int G(\mathbf{r}_0, \mathbf{r}) Q(\mathbf{r}) dV - \int G(\mathbf{r}_0, \mathbf{r}) \nabla \cdot \left(\frac{\mathbf{f}(\mathbf{r})}{\rho(\mathbf{r})} \right) dV. \quad (27)$$

Using Gauss's law, the last integral can be written as $\oint_{\partial V} G(\mathbf{r}_0, \mathbf{r}) \mathbf{f}(\mathbf{r})/\rho(\mathbf{r}) \cdot d\mathbf{S} - \int (\nabla G(\mathbf{r}_0, \mathbf{r})) \cdot (\mathbf{f}(\mathbf{r})/\rho(\mathbf{r})) dV$. When the body force vanishes on the boundary ∂V , the first term in this expression vanishes, and the pressure is given by

$$p(\mathbf{r}_0) = -i\omega \int G(\mathbf{r}_0, \mathbf{r}) Q(\mathbf{r}) dV + \int (\nabla G(\mathbf{r}_0, \mathbf{r})) \cdot \left(\frac{\mathbf{f}(\mathbf{r})}{\rho(\mathbf{r})} \right) dV. \quad (28)$$

7.1. Extracting $G - G^*$ for acoustic waves

We first analyze the expression for $G - G^*$ and take ∂V a large sphere and assume there is no attenuation ($\text{Im}(\kappa) = 0$), and that the field satisfies a radiation boundary condition on ∂V

$$\frac{\partial p}{\partial n} = ikp \quad \text{on } \partial V, \quad (29)$$

where $k = \omega/c$, with c being the wave velocity. The Green's function satisfies the same boundary condition. Using equation (3) with expression (9) then gives the following expression for $G - G^*$:

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\omega \oint_{\partial V} \frac{1}{\rho c} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dS. \quad (30)$$

This well-known equation [28, 33, 45] states that the Green's function can be extracted from field fluctuations excited by sources on a closed surface surrounding \mathbf{r}_A and \mathbf{r}_B .

Let us next consider the case of a closed body where either p or its normal derivative $\partial p/\partial n$ vanishes at the boundary, and consider the Green's function that satisfies the same boundary condition. In that case, expression (3) reduces with equation (9) to

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\omega^2 \int \text{Im}(\kappa(\mathbf{r})) G(\mathbf{r}_A, \mathbf{r}) G^*(\mathbf{r}_B, \mathbf{r}) dV. \quad (31)$$

The imaginary part of the compressibility $\text{Im}(\kappa(\mathbf{r}))$ denotes the local attenuation. Consider the case where random injection sources are present that are spatially uncorrelated and satisfy

$$\langle Q(\mathbf{r}_1) Q^*(\mathbf{r}_2) \rangle = \text{Im}(\kappa(\mathbf{r}_1)) |S(\omega)|^2 \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (32)$$

We assume there are no body forces acting ($\mathbf{f} = 0$), so the response is given by the first term in the right-hand side of equation (28), and the cross-correlation of the corresponding pressure fluctuations satisfies

$$\begin{aligned} \langle p(\mathbf{r}_A) p^*(\mathbf{r}_B) \rangle &= \omega^2 \left\langle \int G(\mathbf{r}_A, \mathbf{r}_1) Q(\mathbf{r}_1) dV_1 \int G^*(\mathbf{r}_B, \mathbf{r}_2) Q^*(\mathbf{r}_2) dV_2 \right\rangle \\ &= \omega^2 \int \int G(\mathbf{r}_A, \mathbf{r}_1) G^*(\mathbf{r}_B, \mathbf{r}_2) \langle Q(\mathbf{r}_1) Q^*(\mathbf{r}_2) \rangle dV_1 dV_2 \\ &= \omega^2 |S(\omega)|^2 \int \text{Im}(\kappa(\mathbf{r})) G(\mathbf{r}_A, \mathbf{r}) G^*(\mathbf{r}_B, \mathbf{r}) dV, \end{aligned} \quad (33)$$

where expression (32) has been used in the last identity. Equation (31) reduces in this case to

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i|S(\omega)|^{-2} \langle p(\mathbf{r}_A) p^*(\mathbf{r}_B) \rangle. \quad (34)$$

In this case, $G - G^*$ follows from field correlations that are excited by volume sources with a strength proportional to the local attenuation rate. Physically, this requirement implies that the energy in the system is balanced because the attenuation is everywhere proportional to the excitation by field fluctuations. This energy balance is required by the fluctuation–dissipation theorem, which is based on thermal equilibrium [2]–[5], [46]. Note that in the absence of dissipation, $\text{Im}(\kappa) = 0$, and expression (31) reduces to $G - G^* = 0$. This result simply states that in this case G is real, but otherwise provides no information about the Green's function. We show in appendix B that the energy of an undamped oscillator that receives random kicks with zero mean grows linearly with time. Such an oscillator therefore cannot be in equilibrium. Similarly, an undamped closed acoustic medium cannot be in equilibrium when excited by random injection rate sources with zero mean. In the absence of equilibrium, one cannot obtain $G - G^*$ from the expectation value of field fluctuations, because the statistical properties of these field fluctuations change with time. In that case one needs to consider $G + G^*$ instead.

7.2. Extracting $G + G^*$ for acoustic waves

We next consider the case when either radiation boundary conditions or homogeneous boundary conditions hold on ∂V . In both cases, equations (4) and (12) give the following expression:

$$\begin{aligned} G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) &= 2 \int_V \left(\frac{1}{\rho} (\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) \right. \\ &\quad \left. - \omega^2 \text{Re}(\kappa(\mathbf{r})) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) \right) dV. \end{aligned} \quad (35)$$

The last term on the right-hand side comes from the contribution of a_2 to expression (4). This term has the same dependence on the Green's function as does the surface integral in expression (30), but now appears in a volume integral.

It is possible to retrieve $G + G^*$ from expression (35) from the following contrived procedure that considers the difference of two states of field fluctuations. Consider first an experiment where only random injection sources excite field fluctuations and that these injection sources are spatially uncorrelated and satisfy

$$\langle Q(\mathbf{r}_1)Q^*(\mathbf{r}_2) \rangle = \text{Re}(\kappa(\mathbf{r}_1)) |S(\omega)|^2 \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (36)$$

Using a derivation analogous to the one used in equation (33), the correlation of the associated pressure fluctuation p^Q is given by

$$\langle p^Q(\mathbf{r}_A)p^{Q*}(\mathbf{r}_B) \rangle = \omega^2 |S(\omega)|^2 \int_V \text{Re}(\kappa(\mathbf{r})) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV. \quad (37)$$

Consider next another experiment where the field fluctuations are excited by uncorrelated body forces that satisfy

$$\langle f_i(\mathbf{r}_1)f_j^*(\mathbf{r}_2) \rangle = \rho(\mathbf{r}_1)|S(\omega)|^2 \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij}, \quad (38)$$

and that there are no injection sources ($Q = 0$). The pressure field p^f for these sources is given by the last term of expression (28), and the correlation of the field fluctuations satisfies

$$\begin{aligned} \langle p^f(\mathbf{r}_A)p^{f*}(\mathbf{r}_B) \rangle &= \left\langle \int (\partial_i G(\mathbf{r}_A, \mathbf{r}_1)) \frac{f_i(\mathbf{r}_1)}{\rho(\mathbf{r}_1)} dV_1 \int (\partial_j G^*(\mathbf{r}_B, \mathbf{r}_2)) \frac{f_j^*(\mathbf{r}_2)}{\rho(\mathbf{r}_2)} dV_2 \right\rangle \\ &= \int \int \frac{(\partial_i G(\mathbf{r}_A, \mathbf{r}_1))(\partial_j G^*(\mathbf{r}_B, \mathbf{r}_2))}{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)} \langle f_i(\mathbf{r}_1)f_j^*(\mathbf{r}_2) \rangle dV_1 dV_2 \\ &= |S(\omega)|^2 \int \frac{1}{\rho} (\nabla G(\mathbf{r}_A, \mathbf{r})) \cdot (\nabla G^*(\mathbf{r}_B, \mathbf{r})) dV, \end{aligned} \quad (39)$$

where expression (38) is used on the last identity. Inserting equations (37) and (39) into equation (35) finally gives

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = \frac{2}{|S(\omega)|^2} (\langle p^f(\mathbf{r}_A)p^{f*}(\mathbf{r}_B) \rangle - \langle p^Q(\mathbf{r}_A)p^{Q*}(\mathbf{r}_B) \rangle). \quad (40)$$

The sum $G + G^*$ can thus be found by measuring the field fluctuations generated by injection sources and body forces and subtracting the cross-correlations of the associated pressure fluctuations. This procedure is, however, usually not practical because one usually does not have the ability to separately record the field fluctuations excited by injection sources and by body forces.

Unfortunately one cannot use expression (35) for the much more practical case where injection sources and body forces act *simultaneously*. Suppose that the injection sources and body forces satisfy equations (36) and (38), and that these different types of sources are mutually uncorrelated: $\langle Q\mathbf{f} \rangle = 0$. Following the steps of the derivation of equations (33) and (39) then gives the sum $\langle p^f(\mathbf{r}_A)p^{f*}(\mathbf{r}_B) \rangle + \langle p^Q(\mathbf{r}_A)p^{Q*}(\mathbf{r}_B) \rangle$ rather than the *difference* that is needed in expression (35). Yet the minus sign in the latter expression is correct, as we verify in appendix C using a normal mode expansion of the Green's function.

Expression (35) can be used to extract $G + G^*$ from a single type of field fluctuations for the special case when the ratio of $\text{Im}(\kappa)$ and $\text{Re}(\kappa)$ does not depend on location:

$$\gamma(\omega) = \frac{\text{Im}(\kappa(\mathbf{r}, \omega))}{\text{Re}(\kappa(\mathbf{r}, \omega))}. \quad (41)$$

In the following, γ may depend on frequency. Since γ is the inverse of the quality factor, condition (41) states that the quality factor does not depend on position. In that case, equation (31) is given by

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\omega^2\gamma \int \text{Re}(\kappa(\mathbf{r})) G(\mathbf{r}_A, \mathbf{r}) G^*(\mathbf{r}_B, \mathbf{r}) dV. \quad (42)$$

This expression can be used to eliminate the last volume integral in equation (35) to give

$$(1 + i\gamma)G(\mathbf{r}_A, \mathbf{r}_B) - (1 - i\gamma)G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\gamma \int_V \frac{1}{\rho} (\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV. \quad (43)$$

Applying the steps leading to equation (39) to the above expression implies that the Green's function can be extracted from field fluctuations extracted by body forces rather than injection sources. Note that in this case G and G^* have different weights, but since G is causal and G^* acausal, this is no restriction. In fact, one can, in principle, determine γ by considering the amplitude ratio of the extracted causal and acausal responses. Note that expression (31) can be used for Green's function extraction from field fluctuations excited by injection sources for an arbitrary attenuation that may be position dependent. In contrast, equation (43) can be used to retrieve the Green's function from field fluctuations excited by body forces, but this expression can only be used when the attenuation, as defined in expression (41), is constant in space.

We use the example of acoustic waves to clarify the name *Lagrangian Green's function extraction* and return to a general compressibility κ . Using the equation of motion for acoustic waves ($-i\omega\rho\mathbf{v} = -\nabla p$, with \mathbf{v} being the particle velocity), one can write expression (35) as

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 4 \int_V \left(\frac{\rho\omega^2}{2} \mathbf{v}^{(G)}(\mathbf{r}, \mathbf{r}_A) \cdot \mathbf{v}^{(G)*}(\mathbf{r}, \mathbf{r}_B) - \frac{\omega^2 \text{Re}(\kappa)}{2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) \right) dV, \quad (44)$$

where $\mathbf{v}^{(G)}(\mathbf{r}, \mathbf{r}')$ is the particle velocity associated with the Green's function $G(\mathbf{r}, \mathbf{r}')$. When $\mathbf{r}_A = \mathbf{r}_B$, the integral on the right-hand side is given by $\int_V (\frac{1}{2}\rho\omega^2 |\mathbf{v}^{(G)}(\mathbf{r}, \mathbf{r}_A)|^2 - (\omega^2 \text{Re}(\kappa)/2) |G(\mathbf{r}, \mathbf{r}_A)|^2) dV$. This difference of kinetic and potential energies is the Lagrangian of the acoustic field [47]. Previous formulations for Green's function extraction were based on energy principles [28, 48], so they were described by a Hamiltonian formulation. As shown in equation (44), the approach taken in this paper is, for acoustic waves, based on the Lagrangian. For this reason, we use the phrase *Lagrangian Green's function extraction*.

One might think that because of equipartitioning of kinetic and potential energy, the right-hand side of equations (35) and (44) vanishes. For an oscillator, the kinetic and potential energies are, however, equal to each other only when the oscillator is driven at resonance (appendix B). We show in expressions (C.6) and (C.8) of appendix C that for a closed system the ratio of the last term of equation (35) to the first term on the right-hand side is equal to ω^2/ω_n^2 for a mode with angular frequency ω_n . The two integrals on the right-hand side of expression (35)

therefore do not, in general, cancel except for an isolated mode with frequency equal to that of the excitation. Since the total response consists of a sum over all modes, the two terms in equation (35) are, in general, not equal to each other and their difference is nonzero.

8. Discussion

Traditionally, the theory of Green's function extraction from field fluctuations is usually presented as if one can either extract $G - G^*$ or $G + G^*$, depending on the chosen convention for the Green's function. We show that for a large class of linear scalar systems one can extract both $G - G^*$ or $G + G^*$ from field fluctuations, and that the requirements for the distribution of the sources of these fluctuations may differ. This extends the type of sources of field fluctuations that can be used for the retrieval of the Green's function and thereby theoretically enlarges the class of systems for which the impulse response can be extracted from field fluctuations. Table 1 summarizes the sources of field fluctuations required for the extraction of $G - G^*$ and $G + G^*$. The applications shown in the right column of table 1 are made possible because of the theory presented in this work.

For systems invariant under time reversal, one needs uncorrelated sources on a closed surface surrounding the receivers to extract $G - G^*$, while one needs sources throughout the volume to extract $G + G^*$. The new expression for $G + G^*$ makes it possible to retrieve the Green's function for such systems from field fluctuations that are excited by volume sources. Because some systems may exhibit only random volume sources instead of surface sources, this new formulation creates new opportunities for Green's function retrieval from field fluctuations.

For static systems, the Green's function is real, so $G - G^*$ is uninformative, and it was not known how to extract the impulse response of such systems from field fluctuations. Using the formulation based on $G + G^*$, we show that for electrostatic systems, the Green's function can, theoretically, be extracted from quasi-static field fluctuations excited by uncorrelated electric dipole sources. This new result may have applications in geophysics where random dipoles can be present due to the charge separation that occurs in fracture processes, streaming potentials and the breakup of menisci during drainage of porous media [42]. It is an open research question under what conditions electrostatics fields can be extracted robustly from naturally occurring field fluctuations.

Just as for systems that are invariant under time reversal, one needs different sources of field fluctuations depending on whether one extracts $G - G^*$ or $G + G^*$ for the diffusion equation. Field fluctuations excited by injection sources (monopoles) allow for the retrieval of $G - G^*$, while field fluctuations generated by current sources (dipoles) make it possible to extract $G + G^*$. The new formalism thus makes it possible to extract the diffusive Green's function from field fluctuations excited by current sources.

For acoustic waves, the new formulation for the retrieval of $G + G^*$ makes it possible to extract the Green's function from the correlation of field fluctuations excited separately from injection sources and body forces. The example of acoustic waves illustrates that the new formalism for Green's function extraction is based on Lagrangian principles instead of Hamiltonian principles.

The possibility to extract $G - G^*$ and $G + G^*$ from field fluctuations gives the principle of Green's function extraction a new degree of freedom that has the potential to open up new applications for the extraction of the system response from field fluctuations because it can accommodate a wider range of sources of field fluctuations than is possible with only the difference $G - G^*$.

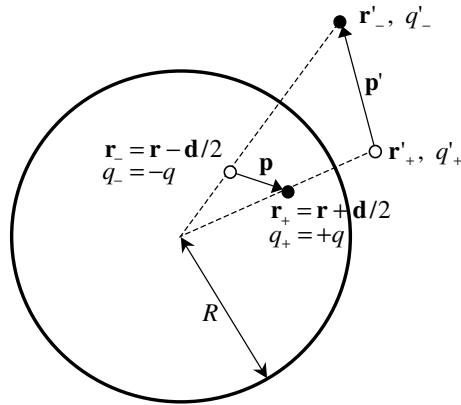


Figure A.1. A dipole \mathbf{p} inside a conducting spherical shell and its image \mathbf{p}' outside the shell.

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Appendix A. The image charges for a dipole in a conducting spherical shell

In this section, we derive the image charges needed to ensure that the electrical potential generated by an electric dipole \mathbf{p} at location \mathbf{r} vanishes at a conducting spherical shell with radius R . The method of images can be used for this problem. As shown in [35], the potential for a charge q at location \mathbf{r} vanishes at the shell when one adds the field generated by a charge q' at a location \mathbf{r}' outside the shell given by

$$q' = -\left(\frac{R}{r}\right)q, \quad \mathbf{r}' = \left(\frac{R}{r}\right)^2 \mathbf{r}. \quad (\text{A.1})$$

We next consider a dipole \mathbf{p} at location \mathbf{r} . As shown in figure A.1, we consider the dipole to consist of the charge $q_+ = q$ at location $\mathbf{r}_+ = \mathbf{r} + \mathbf{d}/2$ and a charge $q_- = -q$ at location $\mathbf{r}_- = \mathbf{r} - \mathbf{d}/2$. Ultimately, we take the limit

$$q \rightarrow \infty, \quad \mathbf{d} \rightarrow 0 \quad \text{such that } \mathbf{p} = q\mathbf{d} \text{ is finite.} \quad (\text{A.2})$$

The required image charges follow by mapping $q_+ = q$ and $q_- = -q$ separately onto image charges following expression (A.1); hence

$$q'_\pm = \mp \frac{R}{|\mathbf{r} \pm \mathbf{d}/2|} q = \mp \frac{R}{r} \left(1 \mp \frac{1}{2} \frac{(\mathbf{r} \cdot \mathbf{d})}{r^2}\right) q + O(qd^2/r^2). \quad (\text{A.3})$$

The locations of these charges are, according to expression (A.1), given by

$$\mathbf{r}'_\pm = \frac{R^2}{|\mathbf{r} \pm \mathbf{d}/2|^2} \left(\mathbf{r} \pm \frac{\mathbf{d}}{2}\right) = \left(\frac{R}{r}\right)^2 \left(\mathbf{r} \pm \frac{\mathbf{d}}{2} \mp (\mathbf{d} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\right) + O(d^2/r). \quad (\text{A.4})$$

The average location of these charges is given by

$$\mathbf{r}' = \frac{\mathbf{r}_+ + \mathbf{r}_-}{2} = \left(\frac{R}{r}\right)^2 \mathbf{r} + O(d^2/r) \rightarrow \left(\frac{R}{r}\right)^2 \mathbf{r}, \quad (\text{A.5})$$

in the limit (A.2). Perhaps surprisingly, the total image charge does not vanish and is in the limit (A.2) given by

$$q' = q'_+ + q'_- = \left(\frac{R}{r}\right) \frac{(\mathbf{d} \cdot \mathbf{r})}{r^2} q + O(qd^2) \rightarrow \frac{R}{r^2} (\mathbf{p} \cdot \hat{\mathbf{r}}). \quad (\text{A.6})$$

The charge difference of the image dipole is given by

$$\frac{q'_+ - q'_-}{2} = - \left(\frac{R}{r}\right) q + O(qd/r), \quad (\text{A.7})$$

while the separation of the image charges is

$$\mathbf{r}'_+ - \mathbf{r}'_- = \left(\frac{R}{r}\right)^2 (\mathbf{d} - 2(\mathbf{d} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}) + O(d^2/r). \quad (\text{A.8})$$

The image dipole vector is thus given by

$$\mathbf{p}' = (\mathbf{r}'_+ - \mathbf{r}'_-) \left(\frac{q'_+ - q'_-}{2}\right) = - \left(\frac{R}{r}\right)^3 (\mathbf{d} - 2(\mathbf{d} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}) q + O(qd^2/r), \quad (\text{A.9})$$

so that in the limit (A.2)

$$\mathbf{p}' = - \left(\frac{R}{r}\right)^3 (\mathbf{p} - 2(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}). \quad (\text{A.10})$$

Note that the image dipole and the original dipole are, in general, not parallel. In fact, these vectors are related by a *Householder transformation* [49].

Appendix B. Properties of an oscillator

Consider a damped oscillator that satisfies

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F}{m}, \quad (\text{B.1})$$

where γ denotes the damping. The Green's function for this system is given by

$$G(t) = \frac{1}{\omega} e^{-\gamma t} \sin(\omega t) H(t), \quad (\text{B.2})$$

with $H(t)$ being the Heaviside function and the angular frequency ω is given by

$$\omega = \sqrt{\omega_0^2 - \gamma^2}. \quad (\text{B.3})$$

We first consider an impulsive random force that kicks the oscillator at time intervals τ :

$$F(t) = \sum_{n=0}^{\infty} F_n \delta(t - t_n), \quad \text{with } t_n = n\tau. \quad (\text{B.4})$$

We assume that the mean of the force vanishes and the forces are uncorrelated

$$\langle F_n \rangle = 0, \quad \langle F_n F_m \rangle = \langle F^2 \rangle \delta_{nm}. \quad (\text{B.5})$$

The response to the excitation is given by

$$x(t) = \sum_{0 < t_n < t} G(t - t_n) \frac{F_n}{m}. \quad (\text{B.6})$$

Since the force has zero mean, the expectation value of the displacement vanishes

$$\langle x(t) \rangle = 0, \quad (\text{B.7})$$

while by virtue of expression (B.5) the variance of the displacement satisfies

$$\langle x^2(t) \rangle = \frac{\langle F^2 \rangle}{m^2} \sum_{0 < n\tau < t} G^2(t - n\tau). \quad (\text{B.8})$$

We assume that τ is much smaller than the period of the oscillator. In that case, the sum can be replaced by an integration over time

$$\langle x^2(t) \rangle = \frac{\langle F^2 \rangle}{m^2 \tau} \int_0^t G^2(t - t') dt'. \quad (\text{B.9})$$

Expression (B.2) can be inserted in the right-hand side, and under the assumption that $\gamma \ll \omega$ one can replace $\sin^2(\omega t) \rightarrow 1/2$; hence

$$\langle x^2(t) \rangle = \frac{\langle F^2 \rangle}{4 m^2 \omega^2 \tau \gamma} (1 - e^{-2\gamma t}). \quad (\text{B.10})$$

For the damped oscillator ($\gamma > 0$), $\langle x^2(t) \rangle$ exponentially reaches the equilibrium value $\langle F^2 \rangle / 4 m^2 \omega^2 \tau \gamma$ as $t \rightarrow \infty$. The response of the undamped oscillator follows by taking the limit $\gamma \rightarrow 0$; in that case, expression (B.10) reduces to

$$\langle x^2(t) \rangle = \frac{\langle F^2 \rangle}{2 m^2 \omega^2 \tau} t. \quad (\text{B.11})$$

The energy of the kicked undamped oscillator thus grows linearly with time.

Next consider the oscillator of expression (B.1) driven by a periodic force

$$F(t) = F_0 \cos(\omega t) = F_0 \text{Re}(e^{-i\omega t}). \quad (\text{B.12})$$

The angular frequency can take any value and is not necessarily given by expression (B.3). The displacement and velocity are given by

$$x(t) = \frac{F_0}{m} \frac{(\omega_0^2 - \omega^2) \cos \omega t + 2\omega\gamma \sin \omega t}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2}, \quad (\text{B.13})$$

$$\dot{x}(t) = \frac{F_0\omega}{m} \frac{-(\omega_0^2 - \omega^2) \sin \omega t + 2\omega\gamma \cos \omega t}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2}. \quad (\text{B.14})$$

The potential energy averaged over one period is

$$\overline{E}_P = \frac{1}{2} m \omega_0^2 \overline{x^2} = \frac{F_0^2}{4m} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2}, \quad (\text{B.15})$$

where the overbar denotes an average over one period. The average kinetic energy is given by

$$\overline{E}_K = \frac{1}{2} m \overline{\dot{x}^2} = \frac{F_0^2}{4m} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2}. \quad (\text{B.16})$$

The ratio of the average kinetic energy and potential energy is given by

$$\frac{\overline{E}_K}{\overline{E}_P} = \frac{\omega^2}{\omega_0^2}. \quad (\text{B.17})$$

Except for the special case when the oscillator is driven at angular frequency $\omega = \omega_0$, the kinetic and potential energies are different, and there is no equipartitioning among kinetic and potential energies. As a consequence, the Lagrangian is nonzero.

Appendix C. Deriving equation (35) from a normal mode expansion

In this appendix, we consider the normal mode expansion of the Green's function for acoustic waves. The modes $u_n(\mathbf{r})$ satisfy equation (25) without an excitation

$$\nabla \cdot \left(\frac{1}{\rho} \nabla u_n \right) + \kappa \omega_n^2 u_n = 0. \quad (\text{C.1})$$

In the absence of attenuation, the modes are real. The orthogonality relation of the modes is obtained by multiplying this expression with a mode u_m , integrating over volume, interchanging n and m and subtracting, applying Gauss' theorem and using that on the boundary either $u = 0$ or $\partial u / \partial n = 0$ [49]. This gives the following orthogonality relation that also defines the normalization of the modes:

$$\int \kappa u_n u_m dV = \delta_{nm}. \quad (\text{C.2})$$

Eliminating κu_n in expression (C.2) using the second term of equation (C.1) and applying Gauss' theorem, one obtains the following alternative orthogonality relation:

$$\int \frac{1}{\rho} (\nabla u_n \cdot \nabla u_m) dV = \omega_n^2 \delta_{nm}. \quad (\text{C.3})$$

The Green's function has the following normal mode expansion [49]:

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{u_n(\mathbf{r}_1) u_n(\mathbf{r}_2)}{\omega_n^2 - \omega^2}. \quad (\text{C.4})$$

Since the modes are real, G is a real function as well.

We next express the integrals on the right-hand side of equation (35) in normal modes. Using the previous expression and the relation $\kappa = 1/\rho c^2$ gives

$$\int_V \frac{\omega^2}{\rho c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV = \sum_{n,m} \frac{\omega^2}{(\omega_n^2 - \omega^2)(\omega_m^2 - \omega^2)} \left(\int \kappa(\mathbf{r}) u_n(\mathbf{r}) u_m(\mathbf{r}) dV \right) u_n(\mathbf{r}_A) u_m(\mathbf{r}_B). \quad (\text{C.5})$$

The orthogonality relation (C.2) reduces the double sum in the right-hand side to a single sum:

$$\int_V \frac{\omega^2}{\rho c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV = \omega^2 \sum_n \frac{u_n(\mathbf{r}_A) u_n(\mathbf{r}_B)}{(\omega_n^2 - \omega^2)^2}. \quad (\text{C.6})$$

This expression differs from the normal mode expansion (C.4) by a factor $\omega^2/(\omega_n^2 - \omega^2)$ for each mode n .

The first term on the right-hand side of expression (35) can also be expressed in the normal mode expansion (C.4):

$$\int_V \frac{1}{\rho} (\nabla G(\mathbf{r}, \mathbf{r}_A)) \cdot (\nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV = \sum_{n,m} \frac{1}{(\omega_n^2 - \omega^2)(\omega_m^2 - \omega^2)} \times \left(\int \frac{1}{\rho(\mathbf{r})} (\nabla u_n(\mathbf{r}) \cdot \nabla u_m(\mathbf{r})) dV \right) u_n(\mathbf{r}_A) u_m(\mathbf{r}_B). \quad (\text{C.7})$$

With the orthogonality relation (C.3), the double sum can be reduced to a single sum over modes

$$\int_V \frac{1}{\rho} (\nabla G(\mathbf{r}, \mathbf{r}_A)) \cdot (\nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV = \sum_n \omega_n^2 \frac{u_n(\mathbf{r}_A) u_n(\mathbf{r}_B)}{(\omega_n^2 - \omega^2)^2}. \quad (\text{C.8})$$

Subtracting equations (C.6) and (C.8) gives

$$\int_V \frac{1}{\rho} \left((\nabla G(\mathbf{r}, \mathbf{r}_A)) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B) - \frac{\omega^2}{c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) \right) dV = \sum_n \frac{u_n(\mathbf{r}_A) u_n(\mathbf{r}_B)}{\omega_n^2 - \omega^2}. \quad (\text{C.9})$$

By virtue of the expansion (C.4), this is equal to the Green's function $G(\mathbf{r}_A, \mathbf{r}_B)$. Since the Green's function is real, expression (C.9) constitutes an alternative proof of equation (35).

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