

## Unified Green's Function Retrieval by Cross Correlation

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It has been shown by many authors that the cross correlation of two recordings of a diffuse wave field at different receivers yields the Green's function between these receivers. Recently the theory has been extended for situations where time-reversal invariance does not hold (e.g., in attenuating media) and where source-receiver reciprocity breaks down (in moving fluids). Here we present a unified theory for Green's function retrieval that captures all these situations and, because of the unified form, readily extends to more complex situations, such as electrokinetic Green's function retrieval in poroelastic or piezoelectric media. The unified theory has a wide range of applications in "remote sensing without a source."

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*Introduction.*—Since the pioneering work of Weaver and Lobkis [1,2], Campillo and Paul [3], and others, the literature on retrieving the acoustic Green's function from the cross correlation of two recordings of a diffuse wave field has expanded spectacularly. Apart from the many successful demonstrations of the method on ultrasonic, geophysical, and oceanographic data, many theoretical developments have been published as well [4–11]. One particular branch of theory is based on the reciprocity principle [12–15]. This theory applies to arbitrary inhomogeneous anisotropic media and therefore not only accounts for the reconstruction of the ballistic wave but also for the primary and multiply scattered waves present in the coda of the Green's function. Recent developments in this branch of research are the extension for situations where time-reversal invariance does not hold (as for electromagnetic waves in conducting media [16–18], acoustic waves in attenuating media [19], or general scalar diffusion phenomena [20]), as well as for situations where source-receiver reciprocity breaks down (as in moving fluids [21,22]). In this Letter we develop a unified representation of Green's functions in terms of cross correlations that covers all these cases. Because of the unified formulation, the theory readily extends to more complex situations, such as electrokinetic Green's function retrieval in poroelastic or piezoelectric media. From this extension it follows, for example, that the cross correlation of passive elastodynamic and electric noise observations at two different receivers yields the elastodynamic response that would be observed at one of the receiver positions as if there were an impulsive electric current source at the other. Hence, cross correlating passive measurements may lead to the remote sensing response of the electrokinetic coupling coefficient, which, in the case of a porous medium, contains relevant information about the permeability of the medium under investigation.

*General matrix-vector equation.*—Diffusion, flow, and wave phenomena can each be captured by the following

differential equation in matrix-vector form [23,24],  $\mathbf{A} \frac{D\mathbf{u}}{Dt} + \mathbf{B}\mathbf{u} + \mathbf{D}_x\mathbf{u} = \mathbf{s}$ , where  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$  is a vector containing space- and time-dependent field quantities,  $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$  is a source vector,  $\mathbf{A} = \mathbf{A}(\mathbf{x})$  and  $\mathbf{B} = \mathbf{B}(\mathbf{x})$  are matrices containing space-dependent material parameters, and  $\mathbf{D}_x$  is a matrix containing the spatial differential operators  $\partial_1$ ,  $\partial_2$ , and  $\partial_3$ .  $D/Dt$  denotes the material time derivative, defined as  $D/Dt = \partial/\partial t + \mathbf{v}^0 \cdot \nabla$ , where  $\partial/\partial t$  is the time derivative in the reference frame and  $\mathbf{v}^0 = \mathbf{v}^0(\mathbf{x})$  the space-dependent flow velocity of the material; the term  $\mathbf{v}^0 \cdot \nabla$  vanishes in nonmoving media. For each application, there exists a real-valued diagonal matrix  $\mathbf{K} = \mathbf{K}^{-1}$  such that  $\mathbf{KAK} = \mathbf{A} = \mathbf{A}^T$ ,  $\mathbf{KBK} = \mathbf{B}^T$  and  $\mathbf{KD}_x\mathbf{K} = -\mathbf{D}_x^T$  (superscript  $T$  denotes transposition).

For mass diffusion of a species through a mixture,  $\mathbf{u}^T = (Y, J_1, J_2, J_3)$  (with  $Y$  denoting the mass fraction of the species and  $J_i$  the mass flux relative to the mixture),  $\mathbf{s}^T = (\dot{\omega}, 0, 0, 0)$  (with  $\dot{\omega}$  the mass production rate),  $\mathbf{A} = \rho \text{diag}(1, 0, 0, 0)$  (with  $\rho$  the mass density),  $\mathbf{B} = \frac{1}{\rho\mathcal{D}} \text{diag}(0, 1, 1, 1)$  (with  $\mathcal{D}$  the diffusion coefficient),  $\mathbf{K} = \text{diag}(1, -1, -1, -1)$  and

$$\mathbf{D}_x = \begin{pmatrix} 0 & \partial_1 & \partial_2 & \partial_3 \\ \partial_1 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 \\ \partial_3 & 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

For other scalar diffusion processes the vectors and matrices are defined in a similar way.

For acoustic wave propagation in a moving attenuating fluid,  $\mathbf{u}^T = (p, v_1, v_2, v_3)$  (with  $p$  the acoustic pressure and  $v_i$  the particle velocity),  $\mathbf{s}^T = (q, f_1, f_2, f_3)$  (with  $q$  the volume injection rate and  $f_i$  the external force),  $\mathbf{A} = \text{diag}(\kappa, \rho, \rho, \rho)$  (with  $\kappa$  the compressibility and  $\rho$  the mass density),  $\mathbf{B} = \text{diag}(b^p, b^v, b^v, b^v)$  (with  $b^p$  and  $b^v$  the loss terms),  $\mathbf{K} = \text{diag}(1, -1, -1, -1)$  and  $\mathbf{D}_x$  again defined by Eq. (1). The spatial variations of the flow velocity  $\mathbf{v}^0$  are assumed small in comparison with those of the particle

velocity of the acoustic wave field (this assumption can be relaxed, but then the equations become more involved [22]).

For electromagnetic diffusion and/or wave propagation in a nonmoving anisotropic medium,  $\mathbf{u}^T = (\mathbf{E}^T, \mathbf{H}^T)$  (with  $\mathbf{E}$  and  $\mathbf{H}$  the electric and magnetic field vectors),  $\mathbf{s}^T = -(\{\mathbf{J}^e\}^T, \{\mathbf{J}^m\}^T)$  (with  $\mathbf{J}^e$  and  $\mathbf{J}^m$  the external electric and magnetic current density vectors),  $\mathbf{A} = \text{block diag}(\boldsymbol{\epsilon}, \boldsymbol{\mu})$  (with  $\boldsymbol{\epsilon}$  and  $\boldsymbol{\mu}$  the permittivity and permeability tensors),  $\mathbf{B} = \text{block diag}(\boldsymbol{\sigma}^e, \boldsymbol{\sigma}^m)$  (with  $\boldsymbol{\sigma}^e$  and  $\boldsymbol{\sigma}^m$  the electric and magnetic conductivity tensors),  $\mathbf{K} = \text{diag}(-1, -1, -1, 1, 1, 1)$  and

$$\mathbf{D}_x = \begin{pmatrix} \mathbf{O} & \mathbf{D}_0^T \\ \mathbf{D}_0 & \mathbf{O} \end{pmatrix}, \quad \mathbf{D}_0 = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}. \quad (2)$$

For elastodynamic wave propagation in a solid,  $\mathbf{u}^T = (\mathbf{v}^T, -\boldsymbol{\tau}_1^T, -\boldsymbol{\tau}_2^T, -\boldsymbol{\tau}_3^T)$  (with  $\mathbf{v}$  and  $\boldsymbol{\tau}_i$  the particle velocity and traction vectors),  $\mathbf{s}^T = (\mathbf{f}^T, \mathbf{h}_1^T, \mathbf{h}_2^T, \mathbf{h}_3^T)$  (with  $\mathbf{f}$  and  $\mathbf{h}_i$  the external force and induced deformation rate vectors), and matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{K}$ , and  $\mathbf{D}_x$  defined in [24].

For electroseismic wave propagation in a saturated porous solid [25,26],  $\mathbf{u}^T = (\mathbf{E}^T, \mathbf{H}^T, \{\mathbf{v}^s\}^T, -\boldsymbol{\tau}_1^T, -\boldsymbol{\tau}_2^T, -\boldsymbol{\tau}_3^T, \mathbf{w}^T, p^f)$  (with  $\mathbf{w} = \varphi(\mathbf{v}^f - \mathbf{v}^s)$  the filtration velocity,  $\varphi$  the porosity, and superscripts  $s$  and  $f$  referring to the solid and fluid phase, respectively),  $\mathbf{s}^T = (-\{\mathbf{J}^e\}^T, -\{\mathbf{J}^m\}^T, \mathbf{f}^T, \mathbf{0}^T, \mathbf{0}^T, \mathbf{0}^T, \{\mathbf{f}^f\}^T, 0)$ , and matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{K}$ , and  $\mathbf{D}_x$  defined in [24]. Omitting  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{J}^e$ , and  $\mathbf{J}^m$  from  $\mathbf{u}$  and  $\mathbf{s}$  gives the field and source vectors for the Biot theory [27]. On the other hand, omitting  $\mathbf{w}$ ,  $p^f$ , and  $\mathbf{f}^f$  and reorganizing  $\mathbf{B}$  results in the electrokinetic equations for a piezoelectric system [28].

In all cases, matrices  $\mathbf{A}(\mathbf{x})$  and  $\mathbf{B}(\mathbf{x})$  can be replaced by convolutional operators  $\mathbf{A}(\mathbf{x}, t)*$  and  $\mathbf{B}(\mathbf{x}, t)*$  to account for more general attenuation mechanisms. We define the Fourier transform of a time-dependent function  $f(t)$  as  $\hat{f}(\omega) = \int f(t) \exp(-j\omega t) dt$ , where  $j$  is the imaginary unit and  $\omega$  denotes the angular frequency. Applying the Fourier transform to all terms in the matrix-vector equation (with  $\mathbf{A}$  and  $\mathbf{B}$  defined as convolutional operators) yields  $\hat{\mathbf{A}}(j\omega + \mathbf{v}^0 \cdot \nabla) \hat{\mathbf{u}} + \hat{\mathbf{B}} \hat{\mathbf{u}} + \mathbf{D}_x \hat{\mathbf{u}} = \hat{\mathbf{s}}$ .

*Reciprocity theorem of the convolution type.*—In general, a reciprocity theorem interrelates two independent states in one and the same domain [29,30]. We consider two independent states that are distinguished by subscripts  $A$  and  $B$ . For an arbitrary spatial domain  $\mathbb{D}$  with boundary  $\partial\mathbb{D}$  and outward pointing normal vector  $\mathbf{n}^T = (n_1, n_2, n_3)$ , the convolution-type reciprocity theorem relating these two states reads [24]

$$\int_{\mathbb{D}} [\hat{\mathbf{u}}_A^T \mathbf{K} \hat{\mathbf{s}}_B - \hat{\mathbf{s}}_A^T \mathbf{K} \hat{\mathbf{u}}_B] d^3\mathbf{x} = \int_{\partial\mathbb{D}} \hat{\mathbf{u}}_A^T \hat{\mathbf{M}}_1 \hat{\mathbf{u}}_B d^2\mathbf{x} + \int_{\mathbb{D}} \hat{\mathbf{u}}_A^T \hat{\mathbf{M}}_2 \hat{\mathbf{u}}_B d^3\mathbf{x}, \quad (3)$$

where  $\hat{\mathbf{M}}_1 = \mathbf{K}\{\mathbf{N}_x - \hat{\mathbf{A}}_A(\mathbf{v}_A^0 \cdot \mathbf{n})\}$  and  $\hat{\mathbf{M}}_2 = \mathbf{K}\{\hat{\mathbf{A}}_B(j\omega + \mathbf{v}_B^0 \cdot \nabla) - \hat{\mathbf{A}}_A(j\omega - \mathbf{v}_A^0 \cdot \nabla) + \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A\}$ , with  $\mathbf{N}_x$  defined

similar as  $\mathbf{D}_x$ , but with  $\partial_i$  replaced by  $n_i$  (hence,  $\mathbf{N}_x$  obeys the symmetry relation  $\mathbf{K}\mathbf{N}_x\mathbf{K} = -\mathbf{N}_x^T$ ). We speak of a convolution-type reciprocity theorem because the multiplications in the frequency domain ( $\hat{\mathbf{u}}_A^T \mathbf{K} \hat{\mathbf{s}}_B$ , etc.) correspond to convolutions in the time domain.

*Green's matrix.*—In state  $A$  we replace the space- and frequency-dependent  $L \times 1$  source vector  $\hat{\mathbf{s}}_A(\mathbf{x}, \omega)$  by a  $L \times L$  frequency-independent point source matrix  $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A)$ , where  $\mathbf{I}$  is the identity matrix. Correspondingly, the  $L \times 1$  field vector  $\hat{\mathbf{u}}_A(\mathbf{x}, \omega)$  is replaced by a  $L \times L$  Green's matrix  $\hat{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A, \omega)$ . For example, the acoustic Green's matrix is given by

$$\hat{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A, \omega) = \begin{pmatrix} \hat{G}^{p,q} & \hat{G}_{,1}^{p,f} & \hat{G}_{,2}^{p,f} & \hat{G}_{,3}^{p,f} \\ \hat{G}_1^{v,q} & \hat{G}_{1,1}^{v,f} & \hat{G}_{1,2}^{v,f} & \hat{G}_{1,3}^{v,f} \\ \hat{G}_2^{v,q} & \hat{G}_{2,1}^{v,f} & \hat{G}_{2,2}^{v,f} & \hat{G}_{2,3}^{v,f} \\ \hat{G}_3^{v,q} & \hat{G}_{3,1}^{v,f} & \hat{G}_{3,2}^{v,f} & \hat{G}_{3,3}^{v,f} \end{pmatrix}(\mathbf{x}, \mathbf{x}_A, \omega). \quad (4)$$

The superscripts refer to the type of observed wave field at  $\mathbf{x}$  and the source type at  $\mathbf{x}_A$ , respectively; the subscripts denote the different components. Note that each column represents a field vector at  $\mathbf{x}$  due to one particular source type at  $\mathbf{x}_A$ .

For state  $B$  we choose the medium parameters identical to those in state  $A$  (i.e.,  $\hat{\mathbf{A}}_B = \hat{\mathbf{A}}_A$ ,  $\hat{\mathbf{B}}_B = \hat{\mathbf{B}}_A$ ) and we choose the flow velocity opposite to that in state  $A$  (i.e.,  $\mathbf{v}_B^0 = -\mathbf{v}_A^0$ ), hence,  $\hat{\mathbf{M}}_2$  vanishes. We replace the source vector  $\hat{\mathbf{s}}_B(\mathbf{x}, \omega)$  and the field vector  $\hat{\mathbf{u}}_B(\mathbf{x}, \omega)$  by  $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_B)$  and  $\hat{\mathbf{G}}_r(\mathbf{x}, \mathbf{x}_B, \omega)$ , respectively, where the subscript  $r$  refers to the reversed flow velocity. With these replacements, Eq. (3) becomes a reciprocity relation for the Green's matrix. The second term on the right-hand side vanishes due to the choice of the opposite flow velocities (flow-reversal theorem [24,31,32]). When we choose  $\mathbf{x}_A$  and  $\mathbf{x}_B$  both in  $\mathbb{D}$  and assume that outside a sphere with finite radius the medium is homogeneous, isotropic and nonflowing, then the boundary integral vanishes as well. This leaves the source-receiver reciprocity relation

$$\mathbf{K} \hat{\mathbf{G}}^T(\mathbf{x}_B, \mathbf{x}_A, \omega) \mathbf{K} = \hat{\mathbf{G}}_r(\mathbf{x}_A, \mathbf{x}_B, \omega). \quad (5)$$

Note that for nonflowing media the subscript  $r$  can be omitted.

*Reciprocity theorem of the correlation type.*—We consider a modified version of the reciprocity theorem. For an arbitrary spatial domain  $\mathbb{D}$  with boundary  $\partial\mathbb{D}$  and outward pointing normal vector  $\mathbf{n}$ , the correlation-type reciprocity theorem reads [24]

$$\int_{\mathbb{D}} [\hat{\mathbf{u}}_A^\dagger \hat{\mathbf{s}}_B + \hat{\mathbf{s}}_A^\dagger \hat{\mathbf{u}}_B] d^3\mathbf{x} = \int_{\partial\mathbb{D}} \hat{\mathbf{u}}_A^\dagger \hat{\mathbf{M}}_3 \hat{\mathbf{u}}_B d^2\mathbf{x} + \int_{\mathbb{D}} \hat{\mathbf{u}}_A^\dagger \hat{\mathbf{M}}_4 \hat{\mathbf{u}}_B d^3\mathbf{x} \quad (6)$$

(superscript  $\dagger$  denotes transposition and complex conjugation), where  $\hat{\mathbf{M}}_3 = \mathbf{N}_x + \hat{\mathbf{A}}_A^\dagger(\mathbf{v}_A^0 \cdot \mathbf{n})$  and

$\hat{\mathbf{M}}_4 = \hat{\mathbf{A}}_B(j\omega + \mathbf{v}_B^0 \cdot \nabla) - \hat{\mathbf{A}}_A^\dagger(j\omega + \mathbf{v}_A^0 \cdot \nabla) + \hat{\mathbf{B}}_B + \hat{\mathbf{B}}_A^\dagger$ . We speak of a correlation-type reciprocity theorem because the multiplications in the frequency domain ( $\hat{\mathbf{u}}_A^\dagger \hat{\mathbf{s}}_B$ , etc.) correspond to correlations in the time domain.

*Green's matrix representation.*—We use Eq. (6) to derive a representation of the Green's matrix in terms of cross correlations. To this end we replace the source vectors again by point source matrices and the field vectors

$$\hat{\mathbf{G}}(\mathbf{x}_B, \mathbf{x}_A, \omega) + \hat{\mathbf{G}}^\dagger(\mathbf{x}_A, \mathbf{x}_B, \omega) = - \oint_{\partial\mathbb{D}} \hat{\mathbf{G}}(\mathbf{x}_B, \mathbf{x}, \omega) \hat{\mathbf{M}}_5 \hat{\mathbf{G}}^\dagger(\mathbf{x}_A, \mathbf{x}, \omega) d^2\mathbf{x} + \int_{\mathbb{D}} \hat{\mathbf{G}}(\mathbf{x}_B, \mathbf{x}, \omega) \hat{\mathbf{M}}_6 \hat{\mathbf{G}}^\dagger(\mathbf{x}_A, \mathbf{x}, \omega) d^3\mathbf{x}, \quad (7)$$

with  $\hat{\mathbf{M}}_5 = \mathbf{N}_x + \hat{\mathbf{A}}^\dagger(\mathbf{v}^0 \cdot \mathbf{n})$  and  $\hat{\mathbf{M}}_6 = -(\nabla \cdot \mathbf{v}^0 - j\omega)2j\Im(\hat{\mathbf{A}}) + \hat{\mathbf{B}} + \hat{\mathbf{B}}^\dagger$ , where  $\nabla$  now acts on the quantity left of it and  $\Im$  denotes the imaginary part. Note that  $\Im(\hat{\mathbf{A}})$  and  $\hat{\mathbf{B}} + \hat{\mathbf{B}}^\dagger$  account for the attenuation of the medium. Since we used Eq. (5), the Green's matrices are now defined in a medium with flow velocity  $+\mathbf{v}^0$  (or zero flow in case of a nonmoving medium). Equation (7) is a general representation of the Green's matrix between  $\mathbf{x}_A$  and  $\mathbf{x}_B$  in terms of cross correlations of observed fields at  $\mathbf{x}_A$  and  $\mathbf{x}_B$  due to sources at  $\mathbf{x}$  on the boundary  $\partial\mathbb{D}$  as well as in the domain  $\mathbb{D}$ . The inverse Fourier transform of the left-hand side is  $\mathbf{G}(\mathbf{x}_B, \mathbf{x}_A, t) + \mathbf{G}^T(\mathbf{x}_A, \mathbf{x}_B, -t)$ , from which  $\mathbf{G}(\mathbf{x}_B, \mathbf{x}_A, t)$  is obtained by taking the causal part. The application of Eq. (7) requires independent measurements of the impulse responses of different types of sources at all  $\mathbf{x} \in \mathbb{D} \cup \partial\mathbb{D}$ . In the following we modify the right-hand side into a direct cross correlation (i.e., without the integrals) of diffuse field observations at  $\mathbf{x}_A$  and  $\mathbf{x}_B$ , the diffusivity being due to a distribution of uncorrelated noise sources. Following Snieder [19] we

$$\langle \hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_B, \omega) \{ \hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_A, \omega) \}^\dagger \rangle = \int_{\mathbb{D}} \hat{\mathbf{G}}(\mathbf{x}_B, \mathbf{x}, \omega) \hat{\boldsymbol{\lambda}}(\mathbf{x}, \omega) \hat{\mathbf{G}}^\dagger(\mathbf{x}_A, \mathbf{x}, \omega) \hat{S}(\omega) d^3\mathbf{x}. \quad (8)$$

Comparing this with the right-hand side of Eq. (7) (with vanishing boundary integral), we obtain

$$\{ \hat{\mathbf{G}}(\mathbf{x}_B, \mathbf{x}_A, \omega) + \hat{\mathbf{G}}^\dagger(\mathbf{x}_A, \mathbf{x}_B, \omega) \} \hat{S}(\omega) = \langle \hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_B, \omega) \{ \hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_A, \omega) \}^\dagger \rangle, \quad (9)$$

assuming  $\hat{\boldsymbol{\lambda}}(\mathbf{x}, \omega) = \hat{\mathbf{M}}_6(\mathbf{x}, \omega)$ . Hence, for those situations in which  $\hat{\mathbf{M}}_6$  is a diagonal matrix with one or more nonzero elements (e.g., for scalar diffusion or acoustic wave propagation in an attenuating medium with either real-valued  $\hat{\mathbf{A}}$  or zero flow velocity  $\mathbf{v}^0$ , for electromagnetic diffusion and/or wave propagation in a nonmoving isotropic attenuating medium, and, under particular conditions, for electrokinetic wave propagation in an isotropic porous or piezoelectric medium [24]), the Green's matrix between  $\mathbf{x}_A$  and  $\mathbf{x}_B$  can be obtained from the cross correlation of observations at those points, assuming that a distribution of uncorrelated noise sources is present in  $\mathbb{D}$ , with excitation function(s) proportional to the local loss function(s) on the diagonal of  $\hat{\mathbf{M}}_6$ . Equation (9) is a generalization of results obtained by Snieder for scalar diffusion [20] and for acoustic wave propagation in an attenuating medium [19].

by Green's matrices. We choose  $\mathbf{x}_A$  and  $\mathbf{x}_B$  again both in  $\mathbb{D}$ , but other choices are relevant as well [16–18]. This time we choose  $\hat{\mathbf{A}}_B = \hat{\mathbf{A}}_A = \hat{\mathbf{A}}$ ,  $\hat{\mathbf{B}}_B = \hat{\mathbf{B}}_A = \hat{\mathbf{B}}$ , and  $\mathbf{v}_B^0 = \mathbf{v}_A^0 = -\mathbf{v}^0$ , so the Green's matrices in both states are defined for the situation of reversed flow. Next we use Eq. (5) as well as the symmetry relations for  $\hat{\mathbf{A}}$ ,  $\hat{\mathbf{B}}$ , and  $\mathbf{N}_x$ . Transposing both sides of the resulting equation yields

separately consider the situation for uncorrelated sources in  $\mathbb{D}$  and on  $\partial\mathbb{D}$ .

*Uncorrelated sources in  $\mathbb{D}$ .*—The boundary integral vanishes when homogeneous boundary conditions apply at  $\partial\mathbb{D}$  or, in the case of infinite  $\mathbb{D}$ , when one or more elements of the loss matrices  $\Im(\hat{\mathbf{A}})$  or  $\hat{\mathbf{B}} + \hat{\mathbf{B}}^\dagger$  are nonzero throughout space. For these situations we consider a noise distribution  $\hat{\mathbf{s}}(\mathbf{x}, \omega)$  throughout  $\mathbb{D}$ , where  $\hat{\mathbf{s}}$  is a vector with elements  $\hat{s}_k$ . We assume that two noise sources  $\hat{s}_k(\mathbf{x}, \omega)$  and  $\hat{s}_l(\mathbf{x}', \omega)$  are mutually uncorrelated for any  $k \neq l$  and  $\mathbf{x} \neq \mathbf{x}'$  in  $\mathbb{D}$ , and that their power spectrum is the same for all  $\mathbf{x}$  and  $k$ , apart from a space- and frequency-dependent excitation function. Hence, we assume that these noise sources obey the relation  $\langle \hat{\mathbf{s}}(\mathbf{x}', \omega) \hat{\mathbf{s}}^\dagger(\mathbf{x}, \omega) \rangle = \hat{\boldsymbol{\lambda}}(\mathbf{x}, \omega) \times \delta(\mathbf{x} - \mathbf{x}') \hat{S}(\omega)$ , where  $\langle \cdot \rangle$  denotes a spatial ensemble average,  $\hat{S}(\omega)$  the power spectrum of the noise, and  $\hat{\boldsymbol{\lambda}}(\mathbf{x}, \omega)$  is a diagonal matrix containing the excitation functions. We express the observed field vector at  $\mathbf{x}_A$  as  $\hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_A, \omega) = \int_{\mathbb{D}} \hat{\mathbf{G}}(\mathbf{x}_A, \mathbf{x}, \omega) \hat{\mathbf{s}}(\mathbf{x}, \omega) d^3\mathbf{x}$  [and a similar expression for  $\hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_B, \omega)$ ]. Evaluating the cross correlation of the observed fields yields

*Uncorrelated sources on  $\partial\mathbb{D}$ .*—When  $\mathbb{D}$  is finite and no homogeneous boundary conditions apply at  $\partial\mathbb{D}$ , the boundary integral in Eq. (7) does not vanish. Assuming the losses in  $\mathbb{D}$  are small, the last integral can be ignored (see [16–18] for a discussion of the effects of ignoring this integral). Hence, under this condition Eq. (7) implies that the Green's matrix between  $\mathbf{x}_A$  and  $\mathbf{x}_B$  can be retrieved from cross correlations of responses of independent impulsive sources on  $\partial\mathbb{D}$  only (note that  $\partial\mathbb{D}$  is not necessarily a closed surface: when the medium is “sufficiently inhomogeneous”  $\partial\mathbb{D}$  can be an open surface [33]). To make Eq. (7) suited for uncorrelated noise sources on  $\partial\mathbb{D}$ , matrix  $\hat{\mathbf{M}}_5$  must be “diagonalized” so that we can follow the same procedure as above. The term  $\hat{\mathbf{A}}^\dagger(\mathbf{v}^0 \cdot \mathbf{n})$  in  $\hat{\mathbf{M}}_5$  is diagonal for scalar diffusion and for acoustic wave propagation in a flowing medium, whereas it vanishes in nonmoving me-

dia. However,  $\mathbf{N}_x$  is not diagonal for any of the discussed applications. Diagonalization of the integral  $-\oint_{\partial\mathbb{D}} \hat{\mathbf{G}}(\mathbf{x}_B, \mathbf{x}, \omega) \mathbf{N}_x \hat{\mathbf{G}}^\dagger(\mathbf{x}_A, \mathbf{x}, \omega) d^2\mathbf{x}$  involves decomposition of the sources at  $\partial\mathbb{D}$  into sources for inward and outward propagating waves. Following the approach discussed in [13,34], assuming  $\partial\mathbb{D}$  is far away from  $\mathbf{x}_A$  and  $\mathbf{x}_B$ , we may approximate the integral (including the minus sign) by  $\oint_{\partial\mathbb{D}} \hat{\mathbf{G}}^\phi(\mathbf{x}_B, \mathbf{x}, \omega) \boldsymbol{\lambda}(\mathbf{x}) \{\hat{\mathbf{G}}^\phi(\mathbf{x}_A, \mathbf{x}, \omega)\}^\dagger d^2\mathbf{x} +$  ghost, where “ghost” refers to spurious events due to cross products of inward and outward propagating waves. When  $\partial\mathbb{D}$  is irregular (which is the case when the sources are randomly distributed) these cross products do not integrate coherently and hence the spurious events are suppressed [35]. When the medium at and outside  $\partial\mathbb{D}$  is homogeneous and isotropic the spurious events are absent. Superscript  $\phi$  refers to new source types at  $\mathbf{x} \in \partial\mathbb{D}$  and  $\boldsymbol{\lambda}(\mathbf{x})$  is a diagonal matrix containing normalization factors. For example, for elastodynamic waves in a solid [34],  $\hat{\mathbf{G}}^\phi(\mathbf{x}_A, \mathbf{x}, \omega)$  is a  $16 \times 4$  matrix, in which the columns represent the elastodynamic wave vectors observed at  $\mathbf{x}_A$  due to  $P$ - and  $S$ -wave sources at  $\mathbf{x}$  (the  $S$ -wave sources with three different polarizations), and the diagonal matrix is defined as  $\boldsymbol{\lambda} = \text{diag}(\frac{2}{\rho c_P}, \frac{2}{\rho c_S}, \frac{2}{\rho c_S}, \frac{2}{\rho c_S})$ , where  $c_P$  and  $c_S$  are the  $P$ - and  $S$ -wave propagation velocities of the medium at and outside  $\partial\mathbb{D}$ . Hence, assuming a distribution of uncorrelated noise sources  $\hat{\mathbf{s}}^\phi(\mathbf{x}, \omega)$  on  $\partial\mathbb{D}$ , we arrive in a similar way as above at Eq. (9), but this time with the observed field vector at  $\mathbf{x}_A$  expressed as  $\hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_A, \omega) = \oint_{\partial\mathbb{D}} \hat{\mathbf{G}}^\phi(\mathbf{x}_A, \mathbf{x}, \omega) \hat{\mathbf{s}}^\phi(\mathbf{x}, \omega) d^2\mathbf{x}$  [and a similar expression for  $\hat{\mathbf{u}}^{\text{obs}}(\mathbf{x}_B, \omega)$ ]. In this form, Eq. (9) is a generalization of [13–22] to all field vectors described earlier. For example, for the electroseismic situation the (9,1)-element of  $\mathbf{G}(\mathbf{x}_B, \mathbf{x}_A, t)$  is the vertical particle velocity of the solid phase at  $\mathbf{x}_B$  due to an impulsive horizontal electric current source at  $\mathbf{x}_A$ . According to Eq. (9) it is retrieved by correlating the 9th element of  $\mathbf{u}^{\text{obs}}(\mathbf{x}_B, t)$ , i.e., the vertical velocity noise field at  $\mathbf{x}_B$ , with the first element of  $\mathbf{u}^{\text{obs}}(\mathbf{x}_A, t)$ , being the horizontal electric noise field at  $\mathbf{x}_A$  (actually a macroscopic sensor measures  $v_3^s + w_3$  [26], so the cross correlation of the measured vertical velocity and horizontal electric noise fields gives the sum of the (9,1) and (21,1) elements of the Green’s matrix).

*Conclusion.*—We have derived a unified representation for Green’s function retrieval by cross correlation, which applies to diffusion phenomena, acoustic waves in flowing attenuating media, electromagnetic diffusion and wave phenomena, elastodynamic waves in anisotropic solids and electrokinetic waves in poroelastic or piezoelectric media. The applications are found in “remote sensing without a source,” which includes observation of parameters such as flow, anelastic loss, and the electrokinetic coupling coefficient.

*Note added in proof.*—In another paper we derive Green’s function representations for higher order linear scalar systems and discuss the connection with energy principles [36].

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