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The influence of topography on the propagation and scattering of surface waves

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The effects of topography on three dimensional surface-wave scattering and surface-wave conversions is treated in the Born approximation. Surface-wave scattering by topography is compared with surface-wave scattering by a mountain root model. The interference effects between surface waves scattered by different parts of a heterogeneity are analysed by considering Fraunhofer diffraction for surface waves. For a smooth heterogeneity a relation is established between the interaction terms and the phase speed derivatives. The partial derivatives of the phase speed c with respect to the topography height h for Love (L) and Rayleigh (R) waves are

$$\left[\frac{1\partial c}{c\partial h}\right]^{\mathrm{L}} = \frac{-1}{4cUI_1}\rho^0 1_1^2(c^2-\beta^2) \quad (z=0)$$

$$\left[\frac{1\partial c}{c\partial h}\right]^{\mathrm{R}} = \frac{-1}{4cUI_1}\rho^0\left[r_2^2c^2 + r_1^2\left(c^2 - 4\left(1-\frac{\beta^2}{\alpha^2}\right)\beta^2\right)\right] \quad (z=0)$$

Phase speed perturbations due to topography can amount to 1–2% and cannot be ignored in surface-wave studies.

1. Introduction

Observations of teleseismic surface waves demonstrate that surface-wave scattering is an important process. Levshin and Berteussen (1979), and Bungum and Capon (1974) showed, using observations from NORSAR, that distinct multipathing of surface waves occurs for periods below 40 s. A formalism to describe the three-dimensional scattering of surface waves by buried heterogeneities was presented in Snieder (1986). (This is referred to as paper 1.)

There is, however, no reason to assume that surface waves are scattered by buried heterogeneities only, since topography variations also cause surface-wave scattering. Even for very idealised models the effects of topography turn out to be very complicated. Asymptotic results for a narrow mountain ridge on a homogeneous two-dimensional half-space are given by Sabina and Willis (1975, 1977). A survey of numerical methods which have been used to study the effects of topography on seismic waves is given by Sanchez-Sesma (1983). Bullit and Toksoz (1985) used ultrasonic Rayleigh waves in an aluminum model to investigate the effects of topography on three-dimensional surface waves. Because of the complexity of the problem this paper is restricted to topography variations that are weak enough to render the Born approximation valid.

The linearised scattering of elastic waves by surface heterogeneities has received considerable interest. The basic theory for this is outlined in Gilbert and Knopoff (1960) for a homogeneous

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half-space, and by Herrera (1964) for a layered medium. Hudson (1977) applied the theory to the generation of the P-wave Coda, while Woodhouse and Dahlen (1978) considered the effect of topography on the free oscillations of the Earth. A completely different approach was used by Steg and Klemens (1974) who analysed Rayleigh waves in solid materials, which they treated as a lattice instead of a continuum.

This paper provides an explicit formalism to analyse three-dimensional surface-wave scattering by topography in a continuous elastic medium. A formalism for the linearised scattering of three-dimensional elastic waves is presented in section 2. It is shown in section 3 how surface-wave scattering by topography can be accommodated in the theory of paper 1. (An appendix is added with a proof that the theory of paper 1 is unaffected if the heterogeneity is nonzero at the surface.) Because of the linearisations these results are only approximations. The validity of the Born approximation is discussed in Hudson and Heritage (1982). In the treatment of the scattering by topography the stress is assumed to behave linearly with depth over the topography. This imposes another restriction on the validity of the results presented here, which is discussed in section 4. The interaction terms due to the topography are analysed in section 5, where they are quantitatively compared with the surface-wave scattering by a mountain-root structure.

The expressions for the scattered surface waves contain integrals over the heterogeneity. Interference effects make the analytic evaluation of these integrals complicated, even for idealised scatterers. In section 6 a formalism is derived for Fraunhofer diffraction by surface waves, which is applied in section 7 to a Gaussian mountain.

It is well known that smooth heterogeneities do not cause surface-wave scattering (Bretherton, 1968), but they do cause variations in the phase speed and the amplitude. In section 8 a heuristic argument is used for the relation between the interaction terms and the phase speed variations. It is shown in section 9 that this leads to the partial derivatives of the phase speed with respect to the density, P-wave speed and S-wave speed as obtained from variational principles (Aki and Richards, 1980). Furthermore, the partial derivatives of the phase speed with respect to the topography are derived.

The results presented here are valid under certain restrictions. Firstly, the heterogeneity must be weak enough to make the Born approximation valid (Hudson and Heritage, 1982), and to allow a linearisation of the stress over the topography height. Secondly, the far field limit is used throughout. Thirdly, a plane geometry is assumed, it is shown in Snieder and Nolet (in preparation) that this condition can easily be relaxed. Fourthly, the slope of the topography has to be small. Lastly, it is assumed that the interaction with the body-wave part of the Green's function can be ignored. Body waves and surface waves are shown to be coupled by strong topography variations in Hudson (1967), Greenfield (1971), Hudson and Boore (1980) or Baumgardt (1985). The locked mode approximation (Harvey, 1981) can in principle be used to take this coupling into account, without using the body-wave Green's function.

Throughout this paper the summation convention is used both for vector or tensor indices, as well as for mode numbers. Vector and tensor components are denoted by Roman subscripts, while Greek indices are used for the mode numbers. The dot product which is used is defined by

$$[\bar{p} \cdot \bar{q}] = p_i^* q_i \tag{1.1}$$

where * denotes complex conjugation.

2. Derivation of the equations for the scattered wave

The equation of motion combined with the elasticity relations can be written as

$$L_{ij} u_j = F_i \tag{2.1}$$

In this expression u_i is the i-th component of the displacement, and F_i is the force which excites the wavefield. The (differential) operator L is defined by

$$L_{ij} = -\rho\omega^2\delta_{ij} - \partial_n c_{inmj} \partial_m \tag{2.2}$$

where $\bar{c}$ is the elasticity tensor. The surface boundary condition is given by

$$n_i \tau_{ij} = 0 \quad \text{at the surface} \tag{2.3}$$

$\bar{n}$ is the normal vector pointing outwards from the medium, and τ_{ij} is the stress tensor

$$\tau_{ij} = c_{ijkl}\partial_k u_l \tag{2.4}$$

It is well known how surface-wave solutions can be obtained from (2.1) and (2.3) if the medium is laterally homogeneous and the surface is flat. Aki and Richards (chapter 7, 1980) treated this problem in great detail. They showed that in that case the solution was given by

$$\bar{u}^0 = G\bar{F} \tag{2.5}$$

which is an abbreviated notation for

$$u_i^0(\bar{r}) = \int G_{ij}(\bar{r}, \bar{r}')F_j(\bar{r}')d^3r' \tag{2.6}$$

The Green's function (G) satisfies

$$L^0_{ij}G_{jk}(\bar{r}, \bar{r}') = \delta_{ik}\delta(\bar{r} - \bar{r}') \tag{2.7}$$

In this expression L^0 is the operator L for a laterally homogeneous medium.

If lateral heterogeneities are present, or if the surface is not flat, scattering of elastic waves occurs. These scattering effects are treated here in a linearised way, i.e., it is assumed that both the lateral inhomogeneities of the medium, and the topography variations are small. In that case the density and the elasticity tensor can be written as

$$\rho(x, y, z) = \rho^0(z) + \epsilon\rho^1(x, y, z) \tag{2.8}$$

$$\bar{c}(x, y, z) = \bar{c}^0(z) + \epsilon c^{-1}(x, y, z) \tag{2.9}$$

The (small) parameter ϵ is added to make explicit that the perturbations are small. Let the topography be given by

$$z = -\epsilon h(x, y) \tag{2.10}$$

The $-$ sign has been added because z is counted positively downward, and h is the topography height above $z = 0$. The functions $\bar{c}^0$ and ρ^0 define together with a zero stress boundary condition at $z = 0$ a laterally homogeneous background medium, which is perturbed by the heterogeneities $\bar{c}^1$ and ρ^1. Since the perturbations are small the wave field can be written as a perturbation series in ϵ

$$\bar{u} = \bar{u}^0 + \epsilon\bar{u}^1 + 0(\epsilon^2) \tag{2.11}$$

In this expression $\bar{u}^1$ denotes the Born approximation to the scattered wave.

Hudson (1977) derived expressions for the scattered wave in the Born approximation. He showed that the wave scattered by the medium heterogeneities (ρ^1 and $\bar{c}^1$), and the topography variation is given by

$$\begin{aligned} u_i^1(\bar{r}) = \Big\{ &+ \int G_{ij}(\bar{r}, \bar{r}')\rho^1(\bar{r}')\omega^2 G_{jl}(\bar{r}', \bar{r}_s)\mathrm{d}V' \\ &- \int \left(\partial_m G_{ij}(\bar{r}, \bar{r}')\right)c^1_{jmnk}(\bar{r}') \\ &\quad \times (\partial_n G_{kl}(\bar{r}', \bar{r}_s))\mathrm{d}V' \\ &+ \int G_{ij}(\bar{r}, \bar{r}')h(\bar{r}')\rho^0(\bar{r}')\omega^2 G_{jl}(\bar{r}', \bar{r}_s)\mathrm{d}S' \\ &- \int \left(\partial_m G_{ij}(\bar{r}, \bar{r}')\right)h(\bar{r}')c^0_{jmnk}(\bar{r}') \\ &\quad \times \left(\partial_n G_{kl}(\bar{r}', \bar{r}_s)\mathrm{d}S'\right)\Big\} F_l(\bar{r}_s) \end{aligned} \tag{2.12}$$

The volume integrals are over the volume of the reference medium ($z > 0$), while the surface integrals are to be evaluated at the surface of the reference medium ($z = 0$). The differentiations are taken with respect to the $\bar{r}'$-coordinates. Hudson (1977) derived this result in the time domain, (2.12) is the same expression in the frequency domain. It has been assumed here that the wave field is excited by a point force $\bar{F}$ in $\bar{r}_s$. A more general excitation can be treated by superposition. It is shown in paper 1 how a moment-tensor excitation can be incorporated.

To derive this result three assumptions have to be made:

(1) the heterogeneity is so weak that multiple scattering can be ignored, i.e., that the Born approximation is valid.

(2) The slope of the heterogeneity has to be small, since in Hudson's derivation it is assumed that $(\nabla h) = 0(\epsilon)$.

(3) The stress should behave linearly over the mountain height, i.e., it is assumed that

$$\tau(-h) = \tau(0) - h\partial_z\tau(0) \tag{2.13}$$

is a good approximation.

Using the Dirac δ-function, we can rewrite (2.12) as

$$u_i^1(\bar{r}) = \int G_{ij}(\bar{r}, \bar{r}')\left[\rho^1 + h\rho^0\delta(z')\right]$$
$$\times\omega^2 G_{jl}(\bar{r}', \bar{r}_s)F_l(\bar{r}_s)\mathrm{d}V'$$
$$-\int\left(\partial_m G_{ij}(\bar{r}, \bar{r}')\right)\left[c^1_{jmnk} + hc^0_{jmnk}\delta(z')\right]$$
$$\times\left(\partial_n G_{kl}(\bar{r}', \bar{r}_s)\right)F_l(\bar{r}_s)\mathrm{d}V' \qquad (2.14)$$

The upshot of this calculation is that topography variations in this approximation act on the scattered waves as if both the mass of the mountain ($h\rho^0$), and the total elasticity of the mountain ($h\bar{c}^0$) are compressed to a δ-function at the surface of the reference medium. For the mass term this is intuitively clear, because for surface waves which penetrate much deeper than the mountain height the precise mass distribution is not very important. For the elasticity term this is less obvious, because it is not clear what the implications are of 'compressing' the total elasticity of the mountain in a δ-function.

3. A formalism for surface wave scattering

Up to this point the theory was developed for an arbitrary elastic medium, and for the complete Green's function. This means that all sorts of complex scattering phenomena can be dealt with. (For example (2.14) could be used to describe the scattering of body waves by anisotropic regions, etc.)

From this point on we restrict ourselves to the surface wave part of the Green's function in an isotropic medium. It is shown in paper 1 that the far field Green's function can conveniently be expressed as a dyad of polarisation vectors. Using

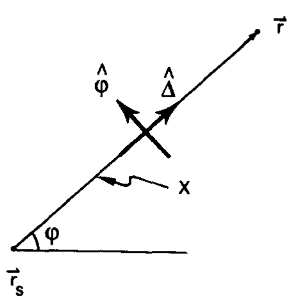

Fig. 1a. Geometry for the direct wave.

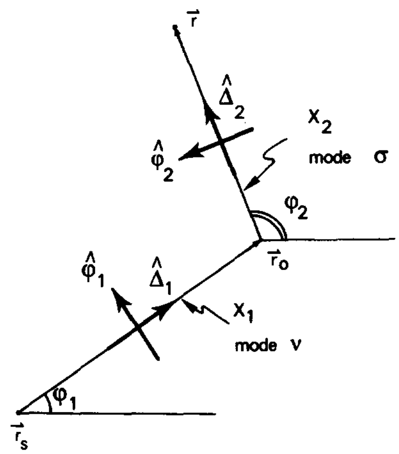

Fig. 1b. Geometry for the scattered wave.

these polarisation vectors we can show that the direct wave is given by

$$\bar{u}^0(\bar{r}) = \bar{p}^\nu(z, \phi)\frac{\mathrm{e}^{i(k_\nu X + \pi/4)}}{\left(\frac{\pi}{2}k_\nu X\right)^{1/2}}\left[\bar{p}^\nu(z_s, \phi).\bar{F}\right] \qquad (3.1)$$

and the scattered wave is for an arbitrary distribution of scatterers

$$\bar{u}^1(\bar{r}) = \int\int \bar{p}^\sigma(z, \phi_2)\frac{\mathrm{e}^{i(k_\sigma X_2 + \pi/4)}}{\left(\frac{\pi}{2}k_\sigma X_2\right)^{1/2}}$$
$$\times V^{\sigma\nu}(x_o, y_o)\frac{\mathrm{e}^{i(k_\nu X_1 + \pi/4)}}{\left(\frac{\pi}{2}k_\nu X_1\right)^{1/2}}$$
$$\times\left[\bar{p}^\nu(z_s, \phi_1).\bar{F}\right]\mathrm{d}x_o\mathrm{d}y_o \qquad (3.2)$$

See Fig. (1a,b) for the definition of variables. Because of the summation convention a double sum over excited modes (ν), and scattered modes (σ) must be applied. The modes are coupled by the interaction matrix $V^{\sigma\nu}$. The only difference with the results in paper 1 is that the depth integrals over the heterogeneity are included in the interaction terms $V^{\sigma\nu}$.

The polarisation vector for Love waves is

$$\bar{p}^\nu(z, \phi) = l_1^\nu(z)\hat{z} \qquad (3.3a)$$

and for Rayleigh waves

$$\bar{p}^{\nu}(z, \phi) = r_1^{\nu}(z)\hat{\Delta} + i r_2^{\nu}(z)\hat{z} \tag{3.3b}$$

Where l_1, r_1 and r_2 are the surface-wave eigenfunctions defined in Aki and Richards (1980). These eigenfunctions are assumed to be normalised according to

$$8 c_{\nu} U_{\nu} I_1^{\nu} = 1 \quad \text{(no summation)} \tag{3.4}$$

In this expression I_1^{ν} is the kinetic energy integral. For Love waves

$$I_1 = 1/2 \int \rho l_1^2 \mathrm{d}z \tag{3.5a}$$

and for Rayleigh waves

$$I_1 = 1/2 \int \rho \left(r_1^2 + r_2^2\right) \mathrm{d}z \tag{3.5b}$$

It was shown in the previous section how surface irregularities could be treated as a δ-function heterogeneity at $z = 0$. Therefore the expression for the interaction coefficients ($V^{\sigma\nu}$) of paper 1 can be used. (In paper 1, (3.2) was derived for buried scatterers. An appendix is added to this paper with a proof that perturbations at the surface do not affect this result.)

Since the scattered wave (2.14) consists of a contribution of the perturbation of the medium parameters, and of a contribution of topography variations, the interaction terms ($V^{\sigma\nu}$) can be decomposed in the following way

$$V^{\sigma\nu} = B^{\sigma\nu} + S^{\sigma\nu} \tag{3.6}$$

$B^{\sigma\nu}$ describes the interaction terms due to the ρ^1 and $\bar{c}^1$ heterogeneity, while $S^{\sigma\nu}$ describes the scattering due to the surface irregularities. The $B^{\sigma\nu}$ terms can be expressed in the surface-wave eigenfunctions l_1, r_1 and r_2. $B_{\mathrm{RL}}^{\sigma\nu}$ is used to denote the scattering from the ν-th Love wave to the σ-th Rayleigh wave by $\bar{c}^1$ and ρ^1, and a similar notation is used for other pairs of interacting modes. It was shown in paper 1 that in this notation the interaction terms for an isotropic medium are given by

$$B_{\mathrm{LL}}^{\sigma\nu} = \int \left[l_1^{\sigma} l_1^{\nu} \rho^1 \omega^2 - (\partial_z l_1^{\sigma})(\partial_z l_1^{\nu}) \mu^1 \right] \mathrm{d}z \cos\phi$$

$$- k_{\sigma} k_{\nu} \int l_1^{\sigma} l_1^{\nu} \mu^1 \mathrm{d}z \cos 2\phi \tag{3.7a}$$

$$B_{\mathrm{RL}}^{\sigma\nu} = \int \left[r_1^{\sigma} l_1^{\nu} \rho^1 \omega^2 + (k_{\sigma} r_2^{\sigma} - \partial_z r_1^{\sigma})(\partial_z l_1^{\nu}) \mu^1 \right] \mathrm{d}z \sin\phi$$

$$- k_{\sigma} k_{\nu} \int r_1^{\sigma} l_1^{\nu} \mu^1 \mathrm{d}z \sin 2\phi \tag{3.7b}$$

$$B_{\mathrm{LR}}^{\sigma\nu} = - B_{\mathrm{RL}}^{\nu\sigma} \tag{3.7c}$$

$$B_{RR}^{\sigma\nu} = \int \left[r_2^{\sigma} r_2^{\nu} \rho^1 \omega^2 - (k_{\sigma} r_1^{\sigma} + \partial_z r_2^{\sigma})(k_{\nu} r_1^{\nu} + \partial_z r_2^{\nu}) \lambda^1 \right.$$

$$\left. - k_{\sigma} k_{\nu} r_1^{\sigma} r_1^{\nu} \mu^1 - 2(\partial_z r_2^{\sigma})(\partial_z r_2^{\nu}) \mu^1 \right] \mathrm{d}z$$

$$+ \int \left[r_1^{\sigma} r_1^{\nu} \rho^1 \omega^2 - (k_{\sigma} r_2^{\sigma} - \partial_z r_1^{\sigma}) \right.$$

$$\left. \times (k_{\nu} r_2^{\nu} - \partial_z r_1^{\nu}) \mu^1 \right] \mathrm{d}z \cos\phi$$

$$- k_{\sigma} k_{\nu} \int r_1^{\sigma} r_1^{\nu} \mu^1 \mathrm{d}z \cos 2\phi \tag{3.7d}$$

In these expressions ∂_z denotes the depth derivative, and ϕ is the scattering angle (Fig. 1b)

$$\phi = \phi_2 - \phi_1 \tag{3.8}$$

Since these relations hold for an isotropic medium the interaction terms are expressed in the perturbations of the Lame parameters (λ^1 and μ^1).

The expressions (3.7a–d) can be used for the calculation of the interaction terms due to topography by substituting

$$\rho^1(x, y, z) \dashrightarrow h(x, y) \rho^0(z) \delta(z)$$

and making the same substitution for λ and μ. In the depth integrals in (3.7a–d) the surface-wave modes then only have to be evaluated at $z = 0$. At that point the vertical derivatives take a particularly simple form. Aki and Richards (1980) showed that at $z = 0$

$$\partial_z l_1 = 0$$

$$\partial_z r_1 = k r_2 \tag{3.9}$$

$$\partial_z r_2 = \frac{-k\lambda^0}{\lambda^0 + 2\mu^0} r_1$$

Using this, the topography interaction terms are given by

$$S_{\mathrm{LL}}^{\sigma\nu} = h\left(l_1^{\sigma} l_1^{\nu} \rho^0 \omega^2 \cos\phi - k_{\sigma} k_{\nu} l_1^{\sigma} l_1^{\nu} \mu^0 \cos 2\phi \right) \tag{3.10a}$$

$$S_{\mathrm{RL}}^{\sigma\nu} = h\left(r_1^{\sigma} l_1^{\nu} \rho^0 \omega^2 \sin\phi - k_{\sigma} k_{\nu} r_1^{\sigma} l_1^{\nu} \mu^0 \sin 2\phi \right) \tag{3.10b}$$

$$S_{\mathrm{LR}}^{\sigma\nu} = -S_{\mathrm{RL}}^{\nu\sigma} \tag{3.10c}$$

$$S_{\mathrm{RR}}^{\sigma\nu} = h\left(r_2^{\sigma} r_2^{\nu} \rho^0 \omega^2 - k_{\sigma} k_{\nu} \mu^0 \frac{3\lambda^0 + 2\mu^0}{\lambda^0 + 2\mu^0} r_1^{\sigma} r_1^{\nu} \right.$$
$$\left. + r_1^{\sigma} r_1^{\nu} \rho^0 \omega^2 \cos\phi - k_{\sigma} k_{\nu} r_1^{\sigma} r_1^{\nu} \mu^0 \cos 2\phi \right) \tag{3.10d}$$

where all quantities have to be evaluated at the surface of the reference medium ($z = 0$).

4. An error analysis of the stress linearisation

The linearisation in the topography in the derivation of Hudson entails two approximations. The Born approximation requires that the scattered wave is sufficiently weak, this is discussed in Hudson and Heritage (1982). The other approximation which is made requires that the stress behaves linearly over the topography height (2.13). An impression of the magnitude of this error can be obtained by verifying this condition for the unperturbed Love waves and Rayleigh waves. This of course gives only a necessary condition for the validity of the stress linearisation, and not a sufficient condition, because the stress in the perturbed medium may behave differently. The error made by the linearisation is defined here as

$$e_i = \frac{\tau_{3i}(z=-h) - (-h)\partial_z \tau_{3i}(z=0)}{\tau_{3i}(z=-h)} \times 100\% \tag{4.1}$$

The eigenfunctions are calculated for the M7 model of Nolet (1977). As a representative example, the error made by linearising τ_{zz} for the fundamental mode as a function of period is shown in Fig. 2 for several values of the topography. The other stress components, and the error for the higher modes behaves similarly. It is quite arbitrary to decide how large an error is acceptable. A relative error of 20% is used here as a maximum since the error made by the stress linearisation is only part of the total error. With this criterium it follows that for a mountain height of 2 km the error is unacceptably large for periods shorter than 12 s. In general, for realistic values of the large scale topography, the linearisation of the stress poses no problems for periods larger than 15 s.

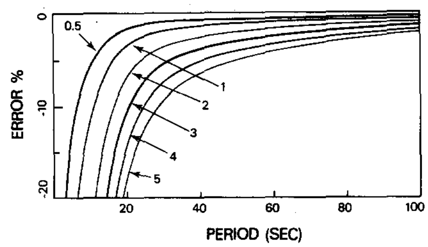

Fig. 2. Relative error for τ_{zz} (defined in (4.1)) for the fundamental Rayleigh mode for several values of the topography height (given in kilometers).

5. The topography interaction terms

In this section the topography interaction terms per unit area ($S^{\sigma\nu}$) are shown for a point topography with a height of 1 km. Since the topography interaction terms are linear in the mountain height (3.10a–d), results for a mountain of arbitrary height can be found by rescaling. The M7 model of Nolet (1977) was used again as a reference medium. The topography interaction terms are a simple function of the scattering angle, and the same convention as in paper 1 is used to de-

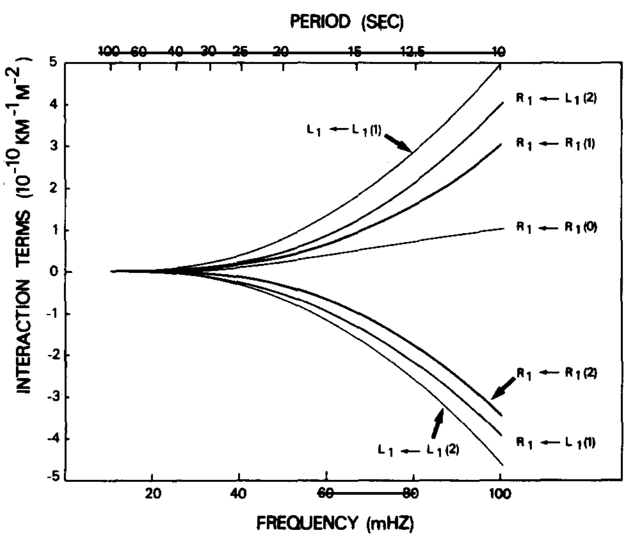

Fig. 3. Topography fundamental mode interaction terms for a mountain height of 1 km.

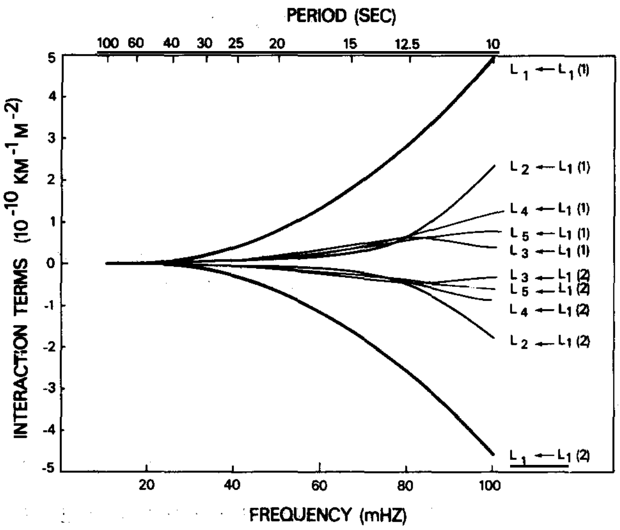

Fig. 4. Topography Love wave interaction terms for a mountain height of 1 km.

note the different azimuth terms. For example, $S_{R_2 \leftarrow L_1}$ (1) denotes the sin ϕ coefficient for the conversion from the fundamental Love mode to the first higher Rayleigh mode, $S_{R_1 \leftarrow R_1}$ (0) indicates the isotropic part of the scattering of the fundamental Rayleigh mode to itself, etc.

Figure 3 shows the different azimuth components of the fundamental mode topography interaction terms. These terms all rapidly increase with frequency. The interaction terms are given in units of (m^{-2}), and should be integrated in (3.2) over the surface of the topography to give the total scattering coefficients.

Just as with surface-wave scattering by a mountain root model (paper 1), the fundamental mode interactions dominate the interactions involving higher modes. As a representative example, the $L_N \leftarrow L_1$ interaction terms are shown in Fig. 4.

It can be seen in Fig. 3 that $S_{L_1 \leftarrow L_1}$ (1) $\approx -S_{L_1 \leftarrow L_1}$ (2), the same holds for the $R_1 \leftarrow R_1$ interactions, and for the $R_1 \leftarrow L_1$ conversion. It turns out that a similar property holds for the interactions with higher modes too. This can be verified in Fig. 4 which shows the $L_N \leftarrow L_1$ topography interaction terms. Therefore, for each conversion $L_N \leftarrow L_1$ the 'cos ϕ' coefficient is almost opposite to the 'cos 2ϕ' coefficient. The reason for this can be seen by rewriting (3.10a–d) in the following way

$$S_{LL}^{\sigma\nu} = hk_\sigma k_\nu l_1^\sigma l_1^\nu \rho^0 \left(c_\sigma c_\nu \cos\phi - \beta^2 \cos 2\phi \right) \tag{5.1a}$$

$$S_{RL}^{\sigma\nu} = hk_\sigma k_\nu r_1^\sigma l_1^\nu \rho^0 \left(c_\sigma c_\nu \sin\phi - \beta^2 \sin 2\phi \right) \tag{5.1b}$$

$$S_{RR}^{\sigma\nu} = S_{RR}^{\sigma\nu}(0) + hk_\sigma k_\nu r_1^\sigma r_1^\nu \rho^0 \left(c_\sigma c_\nu \cos\phi - \beta^2 \cos 2\phi \right) \tag{5.1c}$$

In these expressions c_ν is the phase speed of mode ν, and β is the shear-wave velocity at the surface of the reference medium. For deep modes (long periods) the topography interaction terms are small (Fig. 4), while for shallower modes (shorter periods) the phase speed of both Love and Rayleigh waves is close to the shear-wave velocity in the top layer. This explains that for all cases of importance

$$S(1) \approx -S(2) \tag{5.2}$$

This implies for Love waves

$$S_{LL}^{\sigma\nu} \approx S_{LL}^{\sigma\nu}(1)(\cos\phi - \cos 2\phi) \tag{5.3}$$

which means that the $L \leftarrow L$ radiation pattern has zero's approximately for

$$\phi \approx 0° \quad \text{and} \quad \phi \approx \pm 120° \tag{5.4}$$

so that the scattering in the forward direction is weak, and back-scattering is favoured.

For $R \leftarrow L$ topography scattering, (5.2) implies that

$$S_{RL}^{\sigma\nu} \approx S_{RL}^{\sigma\nu}(1) \sin\phi(1 - 2\cos\phi) \tag{5.5}$$

which means that the radiation pattern for $R \leftarrow L$ conversion by topography has zero's for

$$\phi = 0°, \quad \phi \approx \pm 60° \quad \text{and} \quad \phi = 180° \tag{5.6}$$

For Rayleigh waves a similar analysis cannot be made because of the isotropic term $S_{RR}^{\sigma\nu}(0)$. However, it can be seen in Fig. 3 that (at least for the fundamental mode) this term is relatively small, so that the radiation pattern due to the topography for $R_1 \leftarrow R_1$ scattering is not too different from $L_1 \leftarrow L_1$ scattering.

That this is indeed the case can be verified in

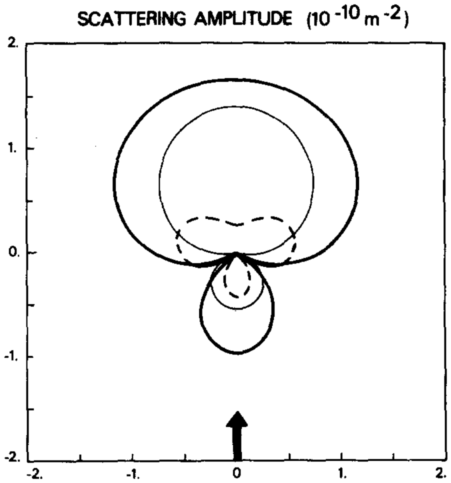

Fig. 5a. Radiation pattern for $R_1 \leftarrow R_1$ scattering for a period of 20 s. Dashed line is scattering by topography of 1 km height, thin line is scattering by a mountain root, thick line is the sum. The direction of the incoming wave is shown by an arrow.

Fig. 5a–c, where the dashed line shows the radiation patterns for the fundamental mode interactions by topography for a period of 20 s. Note that the $L_1 \leftarrow L_1$ topography scattering and the $R_1 \leftarrow R_1$ topography scattering is very weak in the forward direction. The $L_1 \leftarrow L_1$ topography radiation pattern has a zero near $\phi = 120°$, while the $R_1 \leftarrow L_1$ topography radiation pattern has a node near $\phi = 60°$. Observe that the $R_1 \leftarrow R_1$ topogra-

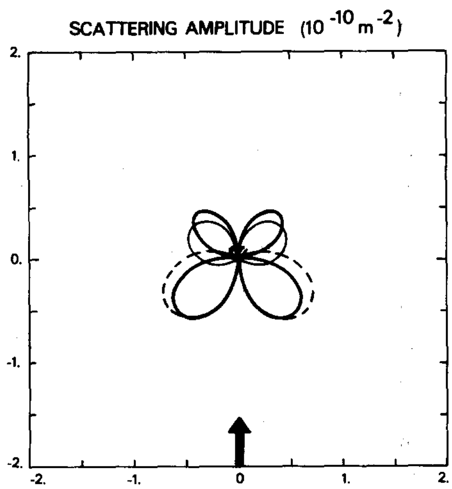

Fig. 5b. Radiation pattern for $L_1 \leftarrow R_1$ scattering for a period of 20 s. Lines defined as in Fig. 5a.

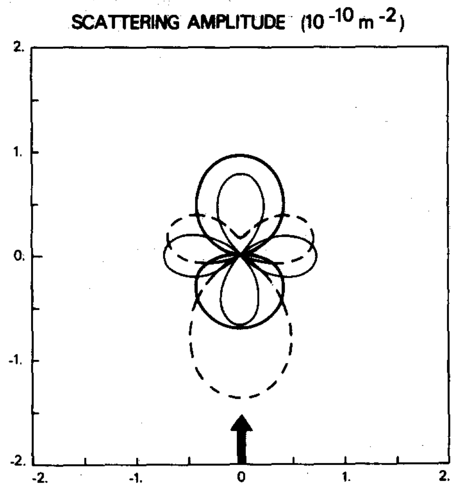

Fig. 5c. Radiation pattern for $L_1 \leftarrow L_1$ scattering for a period of 20 s. Lines defined as in Fig. 5a.

phy radiation pattern differs mostly from the $L_1 \leftarrow L_1$ pattern in the weaker back-scattering. For other periods the topography radiation patterns are very similar because the different azimuth terms behave similarly as a function of frequency.

Figure 5a–c shows the relative importance of scattering by topography to the scattering by buried heterogeneities for a period of 20 s. These figures of course depend strongly on the type of heterogeneity which is considered, on the mountain height, and on frequency. Therefore these figures are only a rough indication of the relative importance in general. In this case a mountain height of 1 km is used, and the mountain root model shown in paper 1 is used for the buried scatterer. (The mountain root model is taken from Mueller and Talwani (1971), and consists of a light, low velocity heterogeneity between 30 and 50 km depth, perturbing the medium in that region with approximately 10%. The structure of a mountain root depends in general both on the height of the mountain, as well as on the horizontal extent. This dependence is ignored here by using the same mountain root model, irrespective of the topography.)

In all three fundamental mode interactions the topography scattering is of the same order of magnitude as the scattering by the mountain root. For periods shorter than 20 s the surface waves are so shallow that the topography scattering tends to dominate. It can be seen in Fig. 5a–c that usually the topography interaction (S), and the mountain root interaction (B) are of the same sign, because the sum of the two terms is larger than each term separately. (The only exception is $L_1 \leftarrow L_1$ scattering at a right angle.) It turns out that this is also the case for the radiation patterns involving higher modes.

That the topography scattering and the scattering by the mountain root enhance each other is caused by the fact that both the topography and the presence of the mountain root give rise to a thickening of the waveguide (the crust). Therefore, these effects are in a sense similar. The difference is that the mountain root heterogeneity results in a perturbation of the medium itself, while the topography affects the surface boundary condition. This gives rise to the different shapes of the radiation patterns, and shows that one should be careful in modelling subsurface heterogeneities with variations of the free surface, as suggested by Bullit and Toksoz (1985).

6. Fraunhofer diffraction of surface waves

The interaction terms which were calculated in the previous section were given per unit area. To obtain the scattered wave (3.2), an integration over the heterogeneity should be performed. A crude estimate of the strength of the scattered wave can be obtained by multiplying the interaction terms with the horizontal extent of the inhomogeneity. This will, however, overestimate the strength of the scattered wave because this procedure ignores interference effects which tend to reduce the scattered wave.

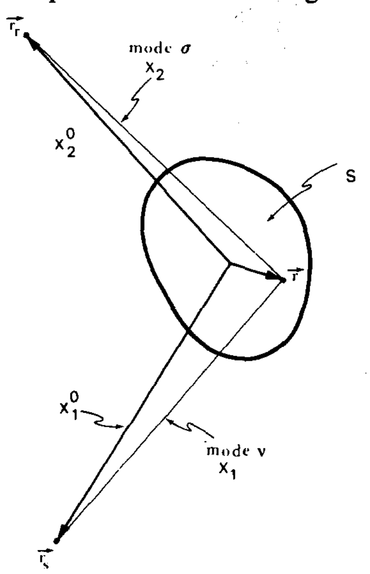

Fig. 6. Geometry for Fraunhofer diffraction.

To incorporate these interference effects, let us consider a localized scatterer which has a horizontal extension which is small compared to the source–scatterer distance, and the scatterer–receiver distance. This means that in the notation of Fig. 6

$$|\bar{r}| \ll X_1^0 \text{ and } |\bar{r}| \ll X_2^0 \tag{6.1}$$

In that case the phase of the integrand in (3.2) can be linearised in $|\bar{r}|$. Furthermore, the variation of the geometrical spreading factors over the scatterer can be ignored, because these variations are of relative order $|\bar{r}|/X^0_{1\text{ or }2}$. In that case the scattered wave can be written as

$$\bar{u}^1(\bar{r}_r) = \bar{p}^\sigma(z_r, \phi_2)\frac{e^{i(k_\sigma X_2^0+\pi/4)}}{\left(\frac{\pi}{2}k_\sigma X_2^0\right)^{1/2}}T^{\sigma\nu}\frac{e^{i(k_\nu X_1^0+\pi/4)}}{\left(\frac{\pi}{2}k_\nu X_1^0\right)^{1/2}} \times\left[\bar{p}^\nu(z_s, \phi_1).\bar{F}\right] \tag{6.2}$$

where $T_{\sigma\nu}$ is the total interaction coefficient

$$T^{\sigma\nu} = \int_S e^{-i((k_\sigma\hat{r}_r + k_\nu\hat{r}_s).\bar{r})}V^{\sigma\nu}(\bar{r})\,dS \tag{6.3}$$

This means that the total interaction term is given by the two-dimensional Fourier transform of the heterogeneity.

The wavenumber of the incoming wave is given by

$$\bar{k}_\nu^{\text{in}} = -\hat{r}_s k_\nu \tag{6.4}$$

and the scattered wave has wavenumber

$$\bar{k}_\sigma^{\text{out}} = \hat{r}_r k_\sigma \tag{6.5}$$

Therefore the Fourier transform (6.3) is to be evaluated at the wavenumber corresponding to the wavenumber change in the scattering event

$$(\Delta\bar{k})^{\sigma\nu} = \bar{k}_\sigma^{\text{out}} - \bar{k}_\nu^{\text{in}} \tag{6.6}$$

The magnitude of this wavenumber can easily be expressed in the scattering angle ϕ

$$\Delta k^{\sigma\nu} = \left(k_\sigma^2 + k_\nu^2 - 2k_\sigma k_\nu \cos\phi\right)^{1/2} \tag{6.7}$$

If the scatterer exhibits cylinder symmetry, the azimuth integration in the Fourier integral can be performed. If one uses the integral representation of the Bessel function it follows that

$$T^{\sigma\nu} = 2\pi \int_0^\infty r J_0(\Delta k^{\sigma\nu} r) V^{\sigma\nu}(r)\,\mathrm{d}r \tag{6.8}$$

So that for a scatterer with cylinder symmetry the total scattering coefficient is just the Fourier–Bessel transform of the heterogeneity.

7. Application to a Gaussian mountain

In this section Fraunhofer diffraction by an idealised Gaussian shaped mountain is considered. This means that it is assumed here that

$$V^{\sigma\nu}(\bar{r}) = V^{\sigma\nu} \exp -r^2/L^2 \tag{7.1}$$

Of course a Gaussian mountain cannot satisfy the conditions (6.1). However, the tail of the scatterer contributes little to the integral, and this error is simply ignored. For a Gaussian mountain the integral (6.8) can be performed analytically. Abramowitz and Stegun (1970) gave an expression for the Fourier–Bessel transform of a Gaussian. Using this result one finds

$$T^{\sigma\nu} = \pi L^2 V^{\sigma\nu} \times \exp\left[-\tfrac{1}{4}\left(k_\sigma^2 + k_\nu^2 - 2k_\sigma k_\nu \cos\phi\right)L^2\right] \tag{7.2}$$

The term $\pi L^2 V^{\sigma\nu}$ is the integral of the heterogeneity over the volume of the scatterer. The exponent term describes the interference effects of different parts of the scatterer. For interactions of the fundamental mode with the higher modes, k_σ and k_ν are different so that the exponent is always negative. This term therefore leads to a weakening of the interactions of the fundamental mode with the higher modes.

It is interesting to consider this interference term in some more detail for unconverted waves

$$k_\sigma = k_\nu = k \tag{7.3}$$

(This condition is almost satisfied for the interaction of the fundamental Love mode with the fundamental Rayleigh mode, since their wavenumbers are usually not too different.) In that case the interference term is given by

$$\exp\left[-\tfrac{1}{4}(kL)^2(1-\cos\phi)\right]$$

If the scatterer is wide compared to the wavelength of the surface wave (i.e., $kL \gg 1$), this term is very small except for $\phi = 0$, so that the radiation pattern is strongly peaked in the forward direction. This effect is known in the theory of scattering of electromagnetic waves as the Mie-effect (Born and Wolf, 1959). In Fig. 7 this effect is shown for $R_1 \leftarrow R_1$ scattering by topography at a period of 20 s. To appreciate the dependence of the shape of the radiation pattern on the width of the mountain, the radiation patterns are normalised. For a small mountain ($L = 0$), the forward scattering is comparable to the back-scattering. As the width of the mountain ($2L$) increases to values comparable to the wavelength of the Rayleigh wave (70 km), the radiation pattern has only one narrow lobe in the forward direction.

The strength of the scattered wave for $R_1 \leftarrow R_1$ scattering at a period of 20 s by topography of 1 km height can be seen in Fig. 8. This figure includes the topography only, the contribution from the mountain root is not taken into account,

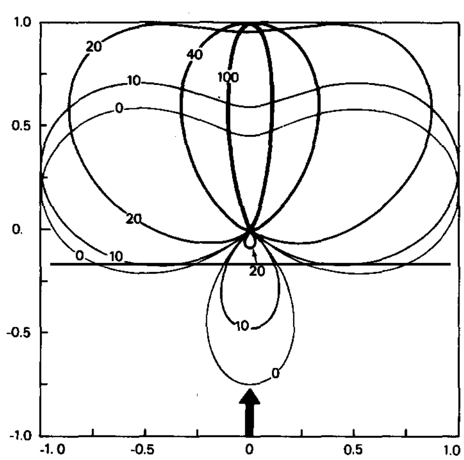

Fig. 7. Normalised topography scattering amplitude $T_{R_1 \leftarrow R_1}$ for a Gaussian mountain for a period of 20 s. Half width L is indicated in kilometers. Incoming wave is shown by an arrow.

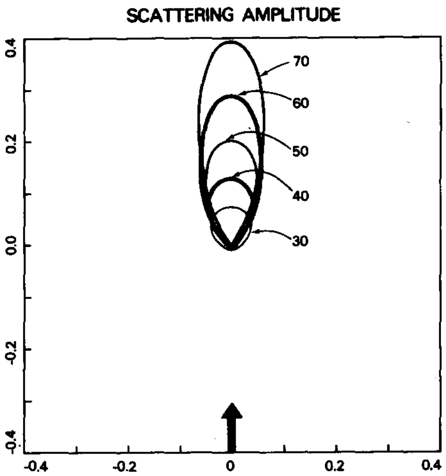

Fig. 8. Topography scattering amplitude $T_{R_1 \leftarrow R_1}$ for a Gaussian mountain of 1 km height at a period of 20 s. Half width L is indicated in kilometers. Incoming wave is shown by an arrow.

because the degree of compensation depends on the size of the mountain too. One should therefore be careful with the interpretation of this figure, since the presence of a mountain root affects the forward scattering drastically (Fig. 5a–c). Furthermore, the strength of the topography scattering depends on the mountain height.

For this particular example it can be concluded that for mountains with a half width less than 30 km the $R_1 \leftarrow R_1$ scattering at 20 s is extremely weak. However, for larger mountains the forward scattering increases rapidly with the mountain size. For mountains with a half width larger than 70 km the total topography interaction coefficient is larger than 0.4. This means that the scattered wave (as it follows from this calculation) is not small compared to the direct wave, which signals the breakdown of the Born approximation. This confirms the NORSAR observations that surface waves with a period shorter than 20 s are strongly scattered (Bungum and Capon, 1974). It will be clear that a mountain complex like the Alps, which has a half width much larger than 70 km, and which has a pronounced root (Mueller and Talwani, 1971) will severely distort the propagation of surface waves with a period shorter than 20 s.

8. Scattering by a band heterogeneity revisited

The perturbation theory derived in this paper and in paper 1 is valid for 'weak inhomogeneities'. The inhomogeneity has to be weak because of the requirement that the scattered waves are small compared to the direct wave. Now suppose we want to apply the theory to a weak and smooth heterogeneity with a large horizontal extent. Smooth means in this context that

$$|\partial_H \mu^1| \ll |k\mu^1| \tag{8.1}$$

where ∂_H is a horizontal derivative, and k is the horizontal wavenumber of the mode under consideration. A similar condition is assumed to hold for λ^1, ρ^1 and h. This condition implies that the heterogeneity varies little on a scale of a horizontal wavelength.

For a heterogeneity with a large horizontal extent, the integrals for the scattered wave may diverge with the size of the heterogeneity, even if the inhomogeneity is relatively weak. This divergence is an effect of the truncation of the perturbation series (Nayfeh, 1973), since the sum of all orders is necessarily finite. Physically this can be understood in the following way. If a wave propagates through a region with a weak and smooth heterogeneity, the only effect of the inhomogeneity is to perturb the local wavenumber. Instead of a solution $\exp(ik_0 x)$ for a laterally homogeneous medium, the laterally heterogeneous

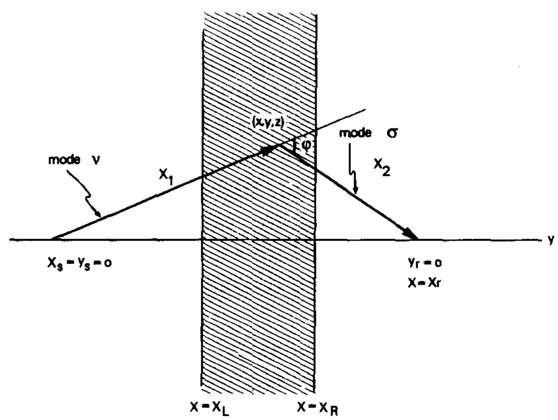

Fig. 9. Geometry for surface wave scattering by a band heterogeneity.

medium has a solution $\exp i\int^x(k_0 + \delta k)dx$. In that case it can be shown with WKBJ theory that reflections and wave conversions are negligible (Bretherton, 1968; Woodhouse, 1974). This means that the Born approximation, which splits the total wave in a direct wave and a scattered wave, does not make much sense physically because in reality there is just one phase shifted transmitted wave.

We discuss this for the band heterogeneity model shown in Fig. 9. It was shown in paper 1 that the total wave in case of propagation through a band heterogeneity is given by

$$\begin{aligned}\bar{u}(\bar{r}_r) = \sum_{\nu} \bar{p}^{\nu}(z_r) \frac{e^{i(k_\nu x_r + \pi/4)}}{\left(\frac{\pi}{2} k_\nu x_r\right)^{1/2}} [\bar{p}^{\nu}(0).\bar{F}] \\ \times \left[1 + \frac{2i\epsilon}{k_\nu} \int_{x_L}^{x_R} V^{\nu\nu}(x)dx\right] \\ + \sum_{\nu \neq \sigma} 2i\epsilon \bar{p}^{\sigma}(z_r) \frac{e^{i(k_\sigma x_r + \pi/4)}}{\left(\frac{\pi}{2} k_\nu k_\sigma\right)^{1/2}} [\bar{p}^{\nu}(0).\bar{F}] \\ \times \int_{x_L}^{x_R} \frac{e^{i(k_\nu - k_\sigma)x}}{\left(k_\nu(x_r - x) + k_\sigma x\right)^{1/2}} V^{\sigma\nu}(x)dx\end{aligned} \tag{8.2}$$

Figure 9 defines the geometric variables in this expression. For convenience the modal summation has for once been made explicit. The unconverted waves are taken together with the direct wave. The last term in (8.2) describes the converted waves. The interaction terms are to be evaluated in the forward direction ($\phi = 0$).

From this point on we shall only concern ourselves with the unconverted wave, since the last integral in (8.2) is negligible for a smooth heterogeneity. (This is because of the oscillation of the exponent term in the integrand.) For simplicity the index ν will be suppressed, but it should be kept in mind that a sum over all unconverted modes is implied.

If the heterogeneity is weak, and not too wide, we can approximate

$$1 + \frac{2i\epsilon}{k} \int V(x)dx = \exp \frac{2i\epsilon}{k} \int V(x)dx \tag{8.3}$$

so that the only effect of the heterogeneity is a phase shift of the transmitted wave.

If the heterogeneity is weak and smooth, but wide, the interval (x_L, x_R) can be divided in thin subintervals. By increasing the number of these subintervals they can be made arbitrarily thin, so that (8.3) can be used for each subinterval. However, the transmission coefficient of a combination of subintervals is in general not related in a simple way to the transmission coefficients of the subintervals.

Rayleigh (1917) addressed this problem by considering the reflection and transmission in a medium consisting of many layers with equal reflection and transmission coefficients. Let r_n and t_n denote the reflection and transmission coefficient of n of these layers. Rayleigh (1917) showed that in that case

$$t_{n+m} = \frac{t_n t_m}{1 - r_n r_m} \tag{8.4}$$

Therefore the transmission coefficient of the combination of two substacks is the product of the transmission coefficient of each substack, provided the reflection coefficients are small.

For a smooth heterogeneity the reflection coefficients are indeed small (Bretherton, 1968), so that the transmission coefficient of a stack of subintervals is the product of the transmission coefficients of each subinterval. Therefore the phase shifts introduced by each subinterval should be added.

This means that, under the restriction that the heterogeneity is smooth, the (divergent) Born approximation should be replaced by

$$1 + \frac{2i\epsilon}{k} \int_{x_L}^{x_R} V(x)dx \dashrightarrow \exp \frac{2i\epsilon}{k} \int_{x_L}^{x_R} V(x)dx \tag{8.5}$$

This renormalisation procedure yields a finite result for a wide and smooth heterogeneity, and is consistent with results from WKBJ theory (Bretherton, 1968). Morse and Feshbach (1953) gave in paragraph 9.3 a rigorous proof of (8.5) for scattering by a potential in the 1-D Schrodinger equation.

9. The partial derivatives of the phase speed with respect to topography

If the expressions (8.2) and (8.5) are combined, the following expression results for the unconverted wave

$$\bar{u}_{\text{unc}}(\bar{r}_r) = \bar{p}(z_r)\frac{\exp i\left[kX + \frac{\pi}{4} + \frac{2}{k}\int_{x_\text{L}}^{x_\text{R}} V(x)\,\mathrm{d}x\right]}{\left(\frac{\pi}{2}kx_r\right)^{1/2}} \times [\bar{p}(0).\bar{F}] \tag{9.1}$$

(The modal summation is not made explicit, and the parameter ϵ is suppressed.) It follows from this expression that the interaction terms V are closely related to the wavenumber perturbation due to the heterogeneity

$$\delta k = \frac{2}{k}V(x) \tag{9.2}$$

and the relative phase speed perturbation is given by

$$\left[\frac{\delta c}{c}\right] = -\frac{2}{k^2}V(x) \tag{9.3}$$

These results are derived for a smooth band heterogeneity. However, since the phase speed depends only on the local properties of the medium, these results can be used for an arbitrary medium with heterogeneities which vary smoothly in the horizontal direction.

The interaction terms V contain a contribution of the buried heterogeneities and a contribution of the topography variations. As an example, consider the phase speed perturbation for Rayleigh waves by a buried heterogeneity. In that case V in (9.3) follows from (3.7d) with $\phi = 0$.

$$\left[\frac{\delta c}{c}\right]^\text{R} = \frac{1}{4cUI_1}\int \mathrm{d}z\big[-\rho^1\omega^2(r_1^2 + r_2^2) + (kr_1 + \partial_z r_2)^2\lambda^1 + \big(2k^2r_1^2 + 2(\partial_z r_2)^2 + (kr_2 - \partial_z r_1)^2\big)\mu^1\big] \tag{9.4}$$

The factor $4cUI_1$ could be added because of the normalisation condition (3.4). Equation 9.4 is equal to expression (7.78) of Aki and Richards (1980), where the Rayleigh-wave phase speed perturbations are calculated with a variational principle. The scattering theory thus produces the same result in a roundabout way, confirming that small variations in the phase speed are treated correctly. For Love waves a similar result can be derived from (3.7a).

Since the interaction terms for topography scattering are known, the partial derivatives of the phase speed with respect to the topography height (h) can be calculated too. For Love waves one finds by inserting (3.10a) (with $\phi = 0$) in (9.3) that

$$\left[\frac{\delta c}{c}\right]^\text{L} = -2\rho^0 l_1^2(c^2 - \beta^2)h \tag{9.5a}$$

while (3.10d) yields for Rayleigh waves

$$\left[\frac{\delta c}{c}\right]^\text{R} = -2\rho^0\left[r_2^2c^2 + r_1^2\left(c^2 - 4\left(1 - \frac{\beta^2}{\alpha^2}\right)\beta^2\right)\right]h \tag{9.5b}$$

In these expressions all variables are to be evaluated at the surface. With the normalisation condition (3.4), this finally gives the partial derivatives of the phase speed with respect of the topography height. For Love waves this leads to

$$\left[\frac{1\partial c}{c\partial h}\right]^\text{L} = \frac{-1}{4cUI_1}\rho^0 l_1^2(c^2 - \beta^2) \quad (z = 0) \tag{9.6a}$$

and for Rayleigh waves

$$\left[\frac{1\partial c}{c\partial h}\right]^\text{R} = \frac{-1}{4cUI_1}\rho^0\left[r_2^2c^2 + r_1^2\left(c^2 - 4\left(1 - \frac{\beta^2}{\alpha^2}\right)\beta^2\right)\right] \quad (z = 0) \tag{9.6b}$$

The partial derivatives of the phase speed with respect to topography height are shown in Fig. 10 for the fundamental modes, calculated with the

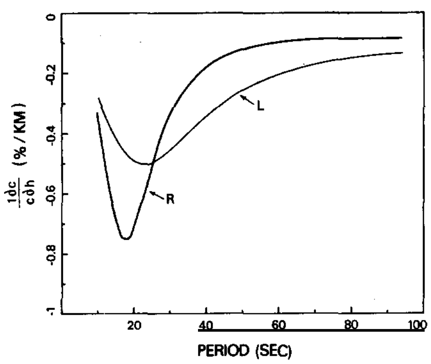

Fig. 10. Phase speed derivative with respect to topography for the fundamental Love mode (L) and for the fundamental Rayleigh mode (R).

M7 model of Nolet (1977). It can be seen that the effect of topography on the phase speed is largest for periods of about 20 s. For large periods the penetration depth of the surface waves is so large that the topography has little effect. For short periods the surface waves only sample the top layer. In that case, a thickening of the top layer by topography does not influence the phase speed.

The relative phase speed perturbation for a topography of 1 km is of the order of 0.5%. This means that for realistic values of the topography (up to several kilometers) this effect cannot be ignored. Since the topography is in general well known, this effect can easily be taken into account in inversions using phase speed observations (Nolet, 1977; Cara et al., 1980; Panza et al., 1980).

10. Summary

Surface-wave scattering by topography can be incorporated in the linearised surface-wave scattering formalism of paper 1. An error analysis shows, however, that for realistic values of the large scale topography (1–5 km) the theory breaks down for periods shorter than 15 s. Furthermore, steep slopes cannot be handled by the theory.

The radiation pattern for scattering by topography shows that the scattering in the forward direction is relatively weak. A comparison with the radiation pattern for a 'mountain root model' shows that scattering by topography, and scattering by a mountain root in general enhance each other. The reason for this is that both effects lead to a thickening of the crustal waveguide.

For scattering by an extended heterogeneity, interference effects between waves radiated from different parts of the scatterer lead to an enhancement of the forward scattering (Mie effect). Furthermore, these interference effects lead to a relative weakening of the interaction of the fundamental mode with the higher modes, compared to the interactions among the fundamental modes.

For a heterogeneity which is smooth in the horizontal direction a relation is established between the interaction terms and the variations in the phase speed. The partial derivatives of the phase speed with respect to the medium parameters, as they are known from variational principles, can be obtained from the scattering theory too. In an analogous way the partial derivatives of the phase speed with respect to topography are obtained.

This is important for the efficient calculation of surface-wave seismograms, and for applying travel-time corrections for the topography. Furthermore, the phase-speed variations due to topography could cause surface-wave focussing and defocussing effects. Ray-tracing techniques, as developed by Gjevik (1974), Babich et al. (1976) or Gaussian beams (Yomogida and Aki, 1985), could be used to investigate this.

The equivalence between topography and surface perturbations of the medium parameters shows that (in this approximation) the inverse problem has a non-unique solution. This poses no problems for the holographic inversion scheme presented in paper 1, since the topography is in general well known. Therefore the surface waves scattered by the topography can be calculated, and subtracted from the recorded surface waves. The remaining scattered surface waves are scattered by the heterogeneity under the topography, so that given enough data the inversion scheme of paper 1 could be used to map the heterogeneity under the topography.

Acknowledgement

I thank Guust Nolet both for providing the context for this research, and for his help and advice.

Appendix

Surface perturbations of medium parameters

Suppose that the density and the elastic parameters are perturbed all the way up to the surface, but that there is a flat topography ($h=0$). As shown in paper 1 the scattered wave satisfies in the interior

$$L^0\bar{u}^1 = -L^1\bar{u}^0 \tag{A1}$$

Linearising the boundary condition (2.3–4) yields for the perturbed quantities at $z = 0$

$$n_i c^0_{ijkl} \partial_k u^1_l = -n_i c^1_{ijkl} \partial_k u^0_l \tag{A2}$$

The r.h.s. of (A1) can be considered as a surface traction exciting the scattered wave. With a representation theorem (Aki and Richards, 1980), (A1) and (A2) can be solved for $\bar{u}^1$

$$u^1_i(\bar{r}) = -\int G_{ij}(\bar{r}, \bar{r}') L^1_{jk}(\bar{r}') u^0_k(\bar{r}') \mathrm{d}V' - \int G_{ij}(\bar{r}, \bar{r}') n^0_m c^1_{mjnk}(\bar{r}') \partial_n u^0_k(\bar{r}') \mathrm{d}S' \tag{A3}$$

Using the representation (2.2) for L^1, (2.6) for the direct wave, and applying a partial integration leads to

$$\begin{aligned} u^1_i(\bar{r}) = &\int G_{ij}(\bar{r}, \bar{r}') \rho^1(\bar{r}') \omega^2 G_{jl}(\bar{r}', \bar{r}_s) F_l(\bar{r}_s) \mathrm{d}V' \\ &- \int \left(\partial_m G_{ij}(\bar{r}, \bar{r}')\right) c^1_{jmnk}(\bar{r}') \\ &\quad \times \left(\partial_n G_{kl}(\bar{r}', \bar{r}_s)\right) F_l(\bar{r}_s) \mathrm{d}V' \\ &+ \int n^0_m G_{ij}(\bar{r}, \bar{r}') c^1_{jmnk}(\bar{r}') \\ &\quad \times \left(\partial_n G_{kl}(\bar{r}', \bar{r}_s)\right) F_l(\bar{r}_s) \mathrm{d}S' \\ &- \int n^0_m G_{ij}(\bar{r}, \bar{r}') c^1_{mjnk}(\bar{r}') \\ &\quad \times \left(\partial_n G_{kl}(\bar{r}', \bar{r}_s)\right) F_l(\bar{r}_s) \mathrm{d}S' \end{aligned} \tag{A4}$$

The third term denotes the 'surface terms' which have been suppressed in paper 1 by considering only buried scatterers. As it turns out, this term is cancelled by the last term in (A4), which is the contribution to the scattered wave from the perturbed boundary conditions (A2). (This follows from the symmetry properties of the elasticity tensor: $c_{jmnk} = c_{mjnk}$.) Therefore only the volume terms in (A4) contribute, and surface perturbations of the medium can be allowed without any modification.

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