Wavefield smoothing and the effect of rough velocity perturbations on arrival times and amplitudes

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SUMMARY

Geometric ray theory is an extremely efficient tool for modelling wave propagation through heterogeneous media. Its use is, however, only justified when the inhomogeneity satisfies certain smoothness criteria. These criteria are often not satisfied, for example in wave propagation through turbulent media. In this paper, the effect of velocity perturbations on the phase and amplitude of transient wavefields is investigated for the situation that the velocity perturbation is not necessarily smooth enough to justify the use of ray theory. It is shown that the phase and amplitude perturbations of transient arrivals can to first order be written as weighted averages of the velocity perturbation over the first Fresnel zone. The resulting averaging integrals are derived for a homogeneous reference medium as well as for inhomogeneous reference media where the equations of dynamic ray tracing need to be invoked. The use of the averaging integrals is illustrated with a numerical example. This example also shows that the derived averaging integrals form a useful starting point for further approximations. The fact that the delay time due to the velocity perturbation can be expressed as a weighted average over the first Fresnel zone explains the success of tomographic inversions schemes that are based on ray theory in situations where ray theory is strictly not justified; in that situation one merely collapses the true sensitivity function over the first Fresnel zone to a line integral along a geometric ray.

Key words: ray theory, scattering, tomography, wave propagation.

1 INTRODUCTION

Wave propagation through complex media is frequently modelled using geometric ray theory. The criteria for the validity of geometric ray theory are that the length-scale of the variations of the medium is much larger than: (a) the wavelength of the employed waves, and (b) the width of the first Fresnel zone (Kravtsov 1988). In many situations these criteria are violated. This is in particular the case for wave propagation through turbulent media such as the ocean or the atmosphere, because the heterogeneity of turbulence is characterized by a power spectrum that contains energy at all wavenumbers. Consider, for example, light propagation through the atmosphere. On the one hand we take it for granted that ray theory can be used to describe the propagation of light through the atmosphere, while on the other hand we know that the blue sky provides direct evidence of the scattering of light by small-scale perturbations that violate the requirements for ray theory. These two notions are clearly contradictory. The use of ray theory for wave propagation through the solid Earth can also be questioned because it has been speculated that the convection in the Earth's mantle can

also have a turbulent character (Yuen *et al.* 1993). Nevertheless, ray theory is extensively used for both forward modelling and inversion of wavefields through media that can be expected to exhibit fluctuations on very short length-scales. Notable examples are ocean tomography and solid Earth traveltime tomography, which rely on ray theory to account for the observed arrival times.

This raises the central question of this study: what properties of the medium determine the arrival time and amplitude of direct-wave arrivals for media in which the requirements for the validity of ray theory are violated? Since it is difficult to provide a general solution to this problem, we restrict ourselves to the case where the perturbations in the medium are weak, but not necessarily smooth. This makes it possible to use the Rytov approximation in Section 2 as a basis for accounting for the perturbation of the wavefield resulting from the perturbation of the medium. The Rytov approximation is derived in Appendix A. For the simplest case of a homogeneous reference medium in two dimensions, it is shown in Section 3 that the phase and amplitude perturbations of the wavefield can be written as weighted averages of the velocity perturbation. The connection with ray theory is established in Appendix B. The

delay time and amplitude perturbation for transient waves are derived in Section 4; it is shown that only the velocity perturbation within the first Fresnel zone contributes to these perturbations. The theory is illustrated with a numerical example in Section 5. In Sections 6 and 7, the extension of the theory to an inhomogeneous smooth reference medium in three dimensions is shown, while in Section 8, the modifications for the 2-D case are presented. The extension to inhomogeneous reference media is complicated because in this situation the relation between the geometrical spreading and the wavefront curvature is non-trivial; this can be handled by using the equations of dynamic ray tracing. In all situations, the delay time and amplitude perturbations can be written as weighted averages of the velocity perturbation over the first Fresnel zone, even when the requirements for ray theory are violated.

2 THE RYTOV APPROXIMATION

In this paper, an averaging integral is derived for the special case of the Helmholtz equation:

$$\nabla^2 u + \frac{\omega^2}{v^2(\mathbf{r})}[1 + n(\mathbf{r})]u = 0, \tag{2.1}$$

where $u(\mathbf{r})$ denotes the wavefield and ω is the angular frequency. It is assumed that a reference velocity $v(\mathbf{r})$ is perturbed by the $n(\mathbf{r})$ term. The important generalization to the Schrödinger equation can be realized by making the following substitutions throughout this paper:

$$\omega^2/v(\mathbf{r}) \rightarrow k^2 (= \text{constant}), \tag{2.2}$$

$$\omega^2 n(\mathbf{r})/v^2(\mathbf{r}) \rightarrow -V(\mathbf{r}), \tag{2.3}$$

where $V(\mathbf{r})$ is the potential and k^2 is the (scaled) constant energy of the particle. The Neumann series solution of (2.1) is given by

$$u(\mathbf{r}) = u_0(\mathbf{r}) + u_B(\mathbf{r}) + O(n^2), \tag{2.4}$$

where $u_0(\mathbf{r})$ is the incident wave and where the Born field $u_B(\mathbf{r})$ equals

$$u_B(\mathbf{r}_0) = -\int \frac{\omega^2}{v^2(\mathbf{r})} G(\mathbf{r}_0, \mathbf{r})n(\mathbf{r})u_0(\mathbf{r}) \, dV. \tag{2.5}$$

Throughout this paper, $G(\mathbf{r}_0, \mathbf{r})$ is the Green's function for the unperturbed medium $v(\mathbf{r})$. It is assumed that the reference medium $v(\mathbf{r})$ is sufficiently smooth to satisfy the requirements for ray theory to be valid. This means that the assumption is made that both the velocity $v(\mathbf{r})$ and the amplitude of the unperturbed Green's function vary little over a wavelength and over the first Fresnel zone. Small-scale components in the velocity model are included in the perturbation $n(\mathbf{r})$.

Truncating the expansion (2.4) after the second term gives the Born approximation. This approximation has the important drawback that the perturbation of the wavefield must be small. This implies in particular that the phase shift generated by the perturbation must be much less than a cycle. For many applications this condition is unacceptably restrictive. As an alternative, the Rytov field has been proposed (Chernov 1960; Rytov, Kravtsov & Tatarskii 1989). It is shown in Appendix A that the Rytov approximation u_R is related to the Born field u_B in the following way:

$$u_R(\mathbf{r}) = u_0(\mathbf{r}) \exp\left(\frac{u_B(\mathbf{r})}{u_0(\mathbf{r})}\right). \tag{2.6}$$

The perturbation of the phase ($\delta\varphi$) and the amplitude (δA) follow immediately from this expression:

$$\delta\varphi = \mathscr{I}m\{u_B/u_0\}, \tag{2.7}$$

$$\delta A = \ln \mathscr{R}e\{u_B/u_0\}. \tag{2.8}$$

Much has been written about the validity of the Rytov approximation (e.g. de Wolf 1965, 1967; Brown 1965, 1967; Heidbreder 1967; Keller 1969; Beydoun & Tarantola 1988). The following two points are important for the purpose of this paper. First, the Rytov approximation does handle large phase shifts. Second, the Rytov approximation does not account for the effects of ray bending on the traveltime, and hence on the phase. The first point follows from the fact that the Rytov approximation accounts only for the first-order perturbation to the phase of the wavefield; see eq. (A10) of Appendix A. This implies that, whenever the phase perturbation is linear in the perturbation, it is accounted for by the Rytov approximation, regardless of whether the phase perturbation is small or not. As an example, consider the Helmholtz equation in one dimension for a constant reference velocity and for the special case that the perturbation $n(z)$ is smooth on the scale of a wavelength. Defining $k_0 = \omega/v$, the Helmholtz equation is given by

$$u_{zz} + k_0^2[1 + n(z)]u = 0. \tag{2.9}$$

Because of the smoothness of $n(z)$, the solution is given by the WKB approximation (Bender & Orszag 1978),

$$u_{WKB}(z_0) = \frac{1}{[1 + n(z_0)]^{1/4}} \exp\left(ik_0 \int_0^{z_0} \sqrt{1 + n(z)} \, dz\right). \tag{2.10}$$

Using the Green's function $G(z_0, z) = (-i/2k_0) \exp(ik_0|z_0 - z|)$, the Rytov solution is obtained using (2.6) and (2.5):

$$u_R(z_0) = e^{ik_0 z_0} \exp\left(\frac{ik_0}{2} \int_0^{z_0} n(z) \, dz\right)$$

$$= \exp\left(ik_0 \int_0^{z_0} \left[1 + \frac{1}{2}n(z)\right] dz\right). \tag{2.11}$$

There are two differences from the WKB solution (2.10). First, the amplitude term $1/[1 + n(z_0)]^{1/4}$ is absent in the Rytov approximation. This term accounts for changes in the amplitude due to the perturbation of the impedance that is associated with the velocity perturbation. This effect is apparently not accounted for by the Rytov approximation. This implies that the amplitude computed with the Rytov approximation is unreliable when the velocity at the observation point is strongly perturbed. Second, the exponents of the WKB solution (2.10) and the Rytov approximation (2.11) are different, but to first order in the perturbation n they are identical. The Rytov approximation does not require the phase shift to be small. As long as $k_0 z_0 n^2 \ll 1$, the phase perturbation is handled well. This implies that the perturbation of the phase ($k_0 z_0 n$) is not required to be small, and that the Rytov approximation breaks down when $z_0 \sim 1/k_0 n^2$. In contrast to this, the Born approximation is only valid when $k_0 z_0 n \ll 1$. This means that for the Born approximation the perturbation of the phase ($k_0 z_0 n$) is required to be small, which implies that this approximation breaks down when $z_0 \sim 1/k_0 n$. Since the perturbation n is assumed to be much smaller than unity, this implies that the Rytov approximation can be used for much greater propagation distances than the Born approximation.

In order to see that the Rytov approximation does not handle ray-bending effects well, consider the multidimensional case where the medium including the perturbation varies smoothly both on the scale of a wavelength and on the scale of the width of the first Fresnel zone. In this case, the ray-geometrical solution holds (Kravtsov 1988), as long as caustics are avoided. The perturbation $n(\mathbf{r})$ causes the rays in the medium to bend. One can show that this leads to changes in the traveltime that are of second order (Snieder & Sambridge 1992, 1993; Snieder & Spencer 1993; Roth, Müller & Snieder 1993; Snieder & Aldridge 1995). Since the Rytov approximation only accounts for first-order changes of the phase, this implies that ray-bending effects are not described by the Rytov approximation. In a similar vein, refracted waves are not accounted for by the Rytov approximation.

Despite this limitation, the Rytov approximation is used to derive an averaging integral for the perturbations of the phase and amplitude. The examples given here show that the first-order perturbations of the phase are accounted for by the Rytov approximation even when these perturbations are not small compared to a cycle. The theory presented here therefore has a direct bearing on linearized tomographic inversions for media that violate the smoothness properties required by ray theory. However, second-order phase perturbations due to ray bending are not taken into account. This implies that the results of this paper only hold for weak perturbations: the non-linear effects of ray bending and multipathing on tomographic inversions are not accounted for.

3 THE AVERAGING THEOREM FOR A PLANE INCOMING WAVE IN TWO DIMENSIONS

In this section, the perturbation of the phase is derived for the simplest case of a homogeneous reference medium in two dimensions. For simplicity it is assumed that the inhomogeneity is only non-zero for $z \geq 0$. The unperturbed wavenumber is given by $k_0 = \omega/v$. The incident wave is a plane wave propagating in the direction of the z-axis:

$$u_0(\mathbf{r}) = \exp i k_0 z. \tag{3.1}$$

The Green's function for the homogeneous reference medium (Morse & Feshbach 1953) is given by

$$G(\mathbf{r}_0, \mathbf{r}) = -\frac{i}{4} H_0^{(1)}(k_0|\mathbf{r}_0 - \mathbf{r}|), \tag{3.2}$$

where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. Using the far-field expression for the Hankel function, one finds that the Green's function in the far field $(k_0|\mathbf{r}_0 - \mathbf{r}| \gg 1)$ is given by

$$G(\mathbf{r}_0, \mathbf{r}) = \frac{-1}{\sqrt{8\pi k_0|\mathbf{r}_0 - \mathbf{r}|}} \exp i\left(k_0|\mathbf{r}_0 - \mathbf{r}| + \frac{\pi}{4}\right). \tag{3.3}$$

In the following, backscattered waves will be ignored since they arrive late and do not contribute to the perturbation of the direct transmitted wave. Limiting the z-integration to the interval between 0 and z_0, the Born field defined in (2.5) is given by

$$u_B(\mathbf{r}_0) = \frac{k_0^2 \exp(i\pi/4)}{\sqrt{8\pi k_0}} \int_0^{z_0} \int_{-\infty}^{\infty} \frac{\exp(ik_0|\mathbf{r}_0 - \mathbf{r}|)n(\mathbf{r})\exp(ik_0 z)}{\sqrt{|\mathbf{r}_0 - \mathbf{r}|}} dx\, dz. \tag{3.4}$$

As a next step, the detour D for a fixed observation point $\mathbf{r}_0$ is defined by

$$D(\mathbf{r}) \equiv |\mathbf{r}_0 - \mathbf{r}| + z - z_0, \tag{3.5}$$

where the geometric variables are defined in Fig. 1. This quantity is the difference between the lengths of the paths travelled by the single scattered wave $(|\mathbf{r}_0 - \mathbf{r}| + z)$ and the direct wave (z_0). Inserting this in eq. (3.4) leads to the following expression for the Born field:

$$u_B(\mathbf{r}_0) = \frac{k_0^2 \exp(i\pi/4)}{\sqrt{8\pi k_0}} \exp(ik_0 z_0) \int \left\{ \int_{-\infty}^{\infty} \frac{n(\mathbf{r})\exp(ik_0 D(\mathbf{r}))}{\sqrt{|\mathbf{r}_0 - \mathbf{r}|}} dx \right\} dz. \tag{3.6}$$

Note that the x-integral resembles a Fourier transform of the perturbation $n(\mathbf{r})$ that is weighted by the geometrical spreading of the scattered wave $(1/\sqrt{|\mathbf{r}_0 - \mathbf{r}|})$. The phase shift $k_0 D(\mathbf{r})$ in expression (3.6) accounts for the delay of the scattered wave compared to the direct wave. It is shown in Section 4 that, when one is only interested in the phase of the first arrival, one can limit the x-integrations to the first Fresnel zone, which is defined here by the condition $k_0 D(\mathbf{r}) < \pi/2$. The following amplitude factor is defined here for later use:

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{1}{\sqrt{|\mathbf{r}_0 - \mathbf{r}|}} \quad \text{(2-D, plane wave)}. \tag{3.7}$$

In general, the perturbation $n(\mathbf{r})$ is not smooth compared to a wavelength and to the width of the first Fresnel zone. This implies that in general the x-integral in (3.6) cannot be solved in the stationary-phase approximation because the term $n(\mathbf{r})$ may exhibit fluctuations over the range where the phase is stationary (the first Fresnel zone). However, the integral $\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp[ik_0 D(\mathbf{r})]\, dx$ can be solved in the stationary-phase approximation. Expanding expression (3.5) to second
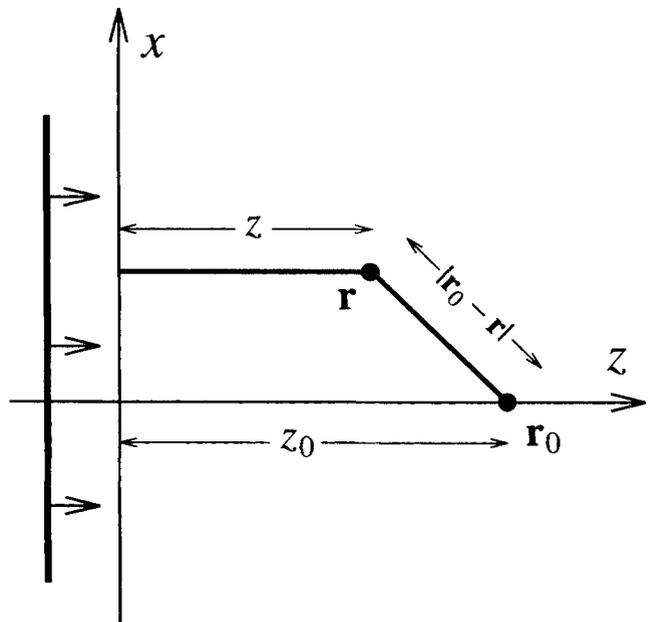

Figure 1. Definition of the geometric variables for a plane incoming wave in two dimensions and a homogeneous reference velocity.

order in x gives

$$D(x, z) = \frac{x^2}{2|z_0 - z|}, \qquad (3.8)$$

so that in the stationary-phase approximation (Bleistein 1984)

$$\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(x, z)) \, dx = F(z, z_0) \sqrt{\frac{2\pi|z_0 - z|}{k_0}} \exp(i\pi/4)$$

$$= \sqrt{\frac{2\pi}{k_0}} \exp(i\pi/4). \qquad (3.9)$$

In this paper, the quantity $F(z, z_0)$ denotes the values of F on the ray in the unperturbed medium: $F(z, z_0) \equiv F(x = 0, z, x_0 = 0, z_0)$. In the last identity of (3.9), the fact that $F(z, z_0) = 1/\sqrt{|z_0 - z|}$ has been used. Note that in the stationary-phase approximation, only the value of F on the reference ray (the z-axis) contributes. In order to recast (3.6) in the form of an averaging integral, divide the x-integral in (3.4) by the left-hand side of (3.9) and multiply the x-integral in (3.4) by the right-hand side of (3.9). This gives, using eq. (3.1) for the direct wave and the definition (3.7):

$$u_{\mathrm{B}}(\mathbf{r}_0) = \frac{ik_0}{2} u_0(\mathbf{r}_0) \int_0^{z_0} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) \exp(ik_0 D(\mathbf{r})) \, dx}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx} \right\} dz . \qquad (3.10)$$

For a homogeneous reference medium this example is easily generalized to other geometries. For the following examples, the Born field can be written in the form of eq. (3.10) with the amplitude factor given by:

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{1}{\sqrt{|\mathbf{r}\|\mathbf{r}_0 - \mathbf{r}|}} \qquad \text{(2-D, point source),} \qquad (3.11)$$

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{1}{|\mathbf{r}_0 - \mathbf{r}|} \qquad \text{(3-D, plane wave),} \qquad (3.12)$$

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{1}{|\mathbf{r}\|\mathbf{r}_0 - \mathbf{r}|} \qquad \text{(3-D, point source),} \qquad (3.13)$$

where it is understood that in three dimensions the x-integrals in (3.10) must be replaced by double integrals over both x and y. When one is interested only in the perturbation of the direct wave, one can limit the integrals over x (and y in three dimensions) to the first Fresnel zone, because waves scattered outside the first Fresnel zone have made such a large detour that they arrive too late to interfere with the direct wave; see Section 4 for further details.

Using (2.7) and (2.8), the first-order perturbation on the phase and the amplitude is given by

$$\delta\varphi(\mathbf{r}_0) = \mathscr{R}e \, \frac{k_0}{2} \int_0^{z_0} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) \exp(ik_0 D(\mathbf{r})) \, dx}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx} \right\} dz , \qquad (3.14)$$

and

$$\delta A = \ln - \mathscr{I}m \, \frac{k_0}{2} \int_0^{z_0} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) \exp(ik_0 D(\mathbf{r})) \, dx}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx} \right\} dz. \qquad (3.15)$$

It can be seen that in expression (3.10) for the Born field and the expressions for the phase and amplitude, a weighted average of the perturbation $n(\mathbf{r})$ is taken over the transverse coordinate. The weight factor follows in a natural way from the scattering integral, and is given by $F(\mathbf{r}, \mathbf{r}_0) \exp ik_0 D(\mathbf{r})$. For this reason, expression (3.10) is referred to as the averaging integral.

As mentioned earlier, the x-integral in expression (3.6) cannot be evaluated in the stationary-phase approximation when $n(\mathbf{r})$ is not smooth. In general, ray theory follows from scattering theory by evaluating the scattering integral in the stationary-phase approximation (e.g. Snieder 1988). This is related to the fact that rays are curves that render the traveltime stationary; in the frequency domain this corresponds to the requirement that the phase is stationary. It is shown in Appendix B, for the special case that the perturbation $n(\mathbf{r})$ is smooth over the first Fresnel zone, that the averaging integrals (3.14) and (3.15) indeed lead to the ray-geometrical perturbations of the phase and amplitude.

4 APPLICATION OF THE AVERAGING THEOREM

For monochromatic signals, one can implement eq. (3.10) or the equivalent expressions (3.14) and (3.15) for the phase and amplitude perturbations in a straightforward fashion. In this application, all integration points (x, z) contribute. The amplitude factor $F(\mathbf{r}, \mathbf{r}_0)$ can vary significantly over the range of integration. This implies that waves scattered at every point (x, z) contribute, and that the contribution depends on the geometry of the incident and scattered wavefield at every location. For monochromatic signals, the averaging integrals of the previous sections are nothing more than a reformulation of the scattering integral.

For transient signals the averaging integrals have more significance. Suppose that one is interested in a transient signal. In such an application, only those points (x, z) that are located within the first Fresnel zone contribute to the transient signal, because the contributions from other regions arrive too late to interfere with the transient arrival. This implies that for transient signals the x-integration can be limited to the first Fresnel zone. This condition limits the domain of x-integration in the averaging integrals. This can be seen by considering the Rytov approximation in the time domain for a band-limited pulse. A transient signal is formed by a Fourier transform to the time domain where frequencies in a band from $\omega_0 - \Delta\omega$ to $\omega_0 + \Delta\omega$ are taken into account. Using (2.6) and (3.10) this quantity can be written as

$$u_{\mathrm{R}}(\mathbf{r}_0, t) = \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} u_0(\mathbf{r}_0, \omega) \exp(-i\omega[t - \Psi(\omega)]) \, d\omega, \qquad (4.1)$$

with

$$\Psi(\omega) = \frac{1}{2v} \int_0^{z_0} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) \exp(ik_0 D(\mathbf{r})) \, dx}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx} \right\} dz. \qquad (4.2)$$

For narrow-band signals, the velocity perturbation leads to a time shift τ and a relative amplitude change ξ of the direct wave; this implies that the direct wave in the Rytov field is approximately given by

$$u_R(\mathbf{r}_0, t) \approx \xi u_0(\mathbf{r}_0, t - \tau), \qquad (4.3)$$

where the time shift τ and the amplitude change ξ still need to be determined. It is shown in Appendix C that the requirement that the approximation (4.3) is satisfied in the least-squares sense leads, for narrow-band signals, to a delay time

$$\tau = \frac{1}{2\Delta\omega} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} \Psi_r(\omega) \, d\omega, \qquad (4.4)$$

where Ψ_r denotes the real part of Ψ.

It should be noted that the analysis of Appendix C breaks down when the bandwidth $\Delta\omega$ is not small. This corresponds to the fact that in general the small-scale velocity perturbations physically lead to dispersion of the direct wave. This implies that in general the assumption (4.3) cannot be applied to broad-band signals since the dispersion caused by the small-scale velocity perturbations leads to changes in the shape of the direct arrival. In the approximation (4.3) some contributions from the original scattering integral have been lost. Only the change in the arrival time and amplitude of the transmitted wave are retained, but truly scattered waves are not taken into account anymore since they have been excluded in the approximation (4.3).

The arguments at the beginning of this section suggest that the contributions around the first Fresnel zone dominate the perturbation arrival time of a transient arrival. It follows from expression (4.4) and the examples shown in Figs 2(a) and (b) that this is indeed the case. To see this, note that the approximation (4.4) and expression (4.2) effectively lead to the following replacement of the averaging function in the averaging integral (3.10):

$$\frac{F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r}))}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx}$$

$$\Rightarrow \frac{1}{2\Delta\omega} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} \frac{F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r}))}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx} \, d\omega. \qquad (4.5)$$

In Figs 2(a) and (b), the real and imaginary parts of the averaging function are shown as a function of transverse distance x (suitably normalized with the curvature D'' of the detour). The thin lines denote the averaging function given by the left-hand side of (4.5) for a number of equidistant frequencies ranging from $\omega_0 - \Delta\omega$ to $\omega_0 + \Delta\omega$. In this example, the value $\Delta\omega/\omega_0 = 0.25$ is used. The thick line is the averaging function for a finite frequency band given by the right-hand side of (4.5). In the first Fresnel zone the averaging functions are in phase, whereas for larger transverse distances the single-frequency averaging functions interfere destructively. This causes the frequency-averaged weight function to decay for


(a) *Real part* / $(\omega_0 D''/c_0)^{1/2}$

(b) *Imaginary part* / $(\omega_0 D''/c_0)^{1/2}$
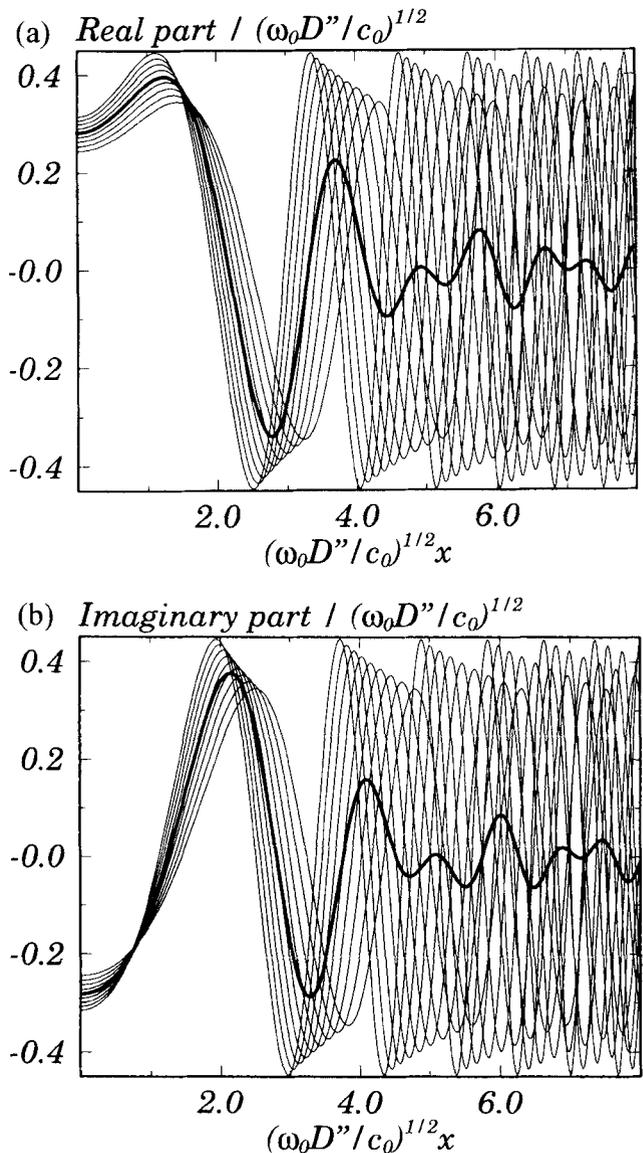
Figure 2. (a) Real part of the averaging function on the left-hand side of (4.5) for 10 different frequencies (thin solid lines). The broad-band weight function defined by the right-hand side of (4.5) is shown by the thick solid line. (b) As (a), but for the imaginary part of the weight function.

transverse distances appreciably larger than the first Fresnel zone. Effectively, this implies that only the contribution of the averaging integrals from the first Fresnel zone contributes to the perturbation of the direct arrival.

Since the detour $D(\mathbf{r})$ is stationary in the first Fresnel zone, $\exp(ik_0 D(\mathbf{r}))$ varies relatively little over the first Fresnel zone. In addition, since ray theory is assumed to hold for the reference medium, the amplitude factor $F(\mathbf{r}, \mathbf{r}_0)$ must vary little over the first Fresnel zone. This suggest that it is a reasonable approximation to make the following replacement in the averaging integrals for transient signals:

$$\frac{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) \exp(ik_0 D(\mathbf{r})) \, dx}{\displaystyle\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r})) \, dx} \Rightarrow \frac{\displaystyle\int_{-x_m}^{x_m} n(\mathbf{r}) W(\mathbf{r}, \mathbf{r}_0) \, dx}{\displaystyle\int_{-x_m}^{x_m} W(\mathbf{r}, \mathbf{r}_0) \, dx}. \qquad (4.6)$$

The finite integration limits $\pm x_m$ reflect the fact that the integration over x for transient signals can be limited to a finite interval, for example the first Fresnel zone, and the original weight function $F(\mathbf{r}, \mathbf{r}_0) \exp(ik_0 D(\mathbf{r}))$ is replaced by a simpler weight function $W(\mathbf{r}, \mathbf{r}_0)$. The denominator $\int W \, dx$ in (4.6) ensures that for a constant perturbation ($n = $ const.) the correct response is obtained. Alternatively, one can argue that the true averaging function $\exp(ik_0 D(\mathbf{r}))$ controls the timing of the scattered waves. However, the waves that are scattered within the first Fresnel zone arrive, by definition, almost in phase. Hence the precise form of the averaging function is not too important, as long as it does not vary strongly over the first Fresnel zone.

The replacement (4.6) was imposed in an *ad hoc* fashion by Groenenboom & Snieder (1995) for a 2-D medium with isotropic point scatterers. Because the exact response could be calculated for such a medium, the accuracy of the replacement (4.6) could be verified. In their results, strong scatterers were used that reduced the amplitude of the direct wave by about a factor of 3. Nevertheless, the amplitudes predicted with an expression similar to eq. (3.15) in combination with the replacement (4.6) agreed very well with the exact response. An additional illustration is given in the numerical example presented in the next section.

5 A NUMERICAL EXAMPLE

In this section, wave propagation through the velocity model of Fig. 3(a) is considered. This model is a realization of a Gaussian random medium as described by Frankel & Clayton (1986). The velocity anomaly has a peak value of about ± 15 per cent. The dominant wavelength of the waves employed is about 200 km, and it follows from Fig. 3(a) that this is comparable to the size of the velocity fluctuations. A plane wave enters the medium from the top of the model and is recorded on a string of receivers indicated by triangles in Fig. 3(a). The detour D, normalized with the background velocity v, for a specific receiver is shown in Fig. 3(b). The first contour line in Fig. 3(b) corresponds to a delay time of half a period. It is clear that the velocity model is not smooth compared to the size of the first Fresnel zone: the velocity model of Fig. 3(a) therefore violates the requirements for the validity of ray theory.

The true wavefield was computed by solving the Helmholtz equation with a finite-difference algorithm. A snapshot of the wavefield just before its arrival at the receivers is shown in Fig. 3(c). As a source wavelet, a cosine modulated with an exponential has been used:

$$S(t) = \exp\left(-\left(\frac{2\pi t}{\alpha T_0}\right)^2\right) \cos\left(\frac{2\pi t}{T_0}\right), \qquad (5.1)$$

where the values $T_0 = 50$ s, $\alpha = 5$ have been used. From the finite-difference seismograms, the arrival time and amplitude of the first arrival were picked by locating the position and magnitude of the first maximum in the wavefield. The resulting arrival time and amplitude are shown as the thick solid lines in Figs 4 and 5. In addition, in Figs 4(a) and 5(a) the ray-geometrical arrival time is shown by the thin solid lines. (The ray-geometrical arrival time was computed by integrating the slowness anomaly over the straight rays of the homogeneous reference medium; it is thus the arrival time predicted by first-order ray theory.) It can be seen in Fig. 4(a) that, although the

ray-geometrical arrival time follows the trend in the true arrival times, it exhibits oscillations that are not present in the true arrival time. This is due to the fact that ray theory does not account for the smoothing properties of a wavefield that are associated with a finite wavelength.

A first example of the arrival-time and amplitude changes predicted by eqs (3.14) and (3.15) with the replacement (4.6) for the averaging integral is shown in Figs 4(a) and (b), with the weight function W defined by the exponential $\exp(ikD)$ modulated with the envelope of the source wavelet (5.1):

$$W(\mathbf{r}, \mathbf{r}_0) = \exp(ikD(\mathbf{r}, \mathbf{r}_0)) \exp\left(-\left(\frac{2\pi D(\mathbf{r}, \mathbf{r}_0)}{\alpha T_0 v}\right)^2\right). \qquad (5.2)$$

The detour $D(\mathbf{r}, \mathbf{r}_0)$ is normalized with the reference velocity v to give a delay time. It can be seen from Fig. 4(a) that the traveltime perturbations thus predicted agree quite well with the true traveltimes. In any case, an agreement with the true arrival times is obtained that lacks the spurious oscillations that are predicted by the ray-geometrical traveltimes (shown by a thin line). Note that the amplitudes shown in Fig. 4(b) are not so well predicted by the averaging integrals: the regions of focusing and defocusing are well predicted, but the amplitudes in the focusing regions are underestimated by the averaging integrals. One should note, however, that the amplitude anomalies are about 100 per cent, so it is not surprising that a first-order theory is not very accurate. In addition, it is well known that modest changes in the velocity model can lead to drastic changes in the amplitudes (White, Nair & Bayliss 1988). This reflects the fact that the dynamic properties of a wavefield are much more sensitive to changes in the velocity model than the kinematic properties of a wavefield. It should be noted that neglecting the variation of the amplitude factor $F(\mathbf{r}, \mathbf{r}_0)$ over the first Fresnel zone is of little consequence because this factor varies by less than 10 per cent over the first Fresnel zone, hence it cannot be the main cause of the discrepancies in the amplitude variations.

When the modulation with the source wavelet is left out in the averaging function (5.2), one obtains arrival times from the averaging integral (3.14) that are much less accurate than the arrival times shown by the dashed line in Fig. 4(a). In that situation the averaging integrals lead to errors in the arrival times that are comparable in size to the errors that are produced by first-order ray theory. This means that making the averaging function W in (4.6) more localized around the Fresnel zone leads to improved estimates of the arrival time of the direct wave. This is consistent with the results of Section 4; the arrival time of a transient arrival is, to first order, only influenced by the slowness perturbations in the first Fresnel zone.

It turns out that the details of the choice of the averaging function W in (4.6) are not very important. This is illustrated in Fig. 5(a), where the arrival time and the amplitude are shown when the weight function $W = 1$ has been used. In this example, the x-integrals in (4.6) are truncated at a value x_m that corresponds to a detour of one-quarter of a wavelength: $kD(x_m) = \pi/2$. It can be seen that the arrival times predicted by this averaging function (the dashed line in Fig. 5a) match the true arrival times as well as the arrival times obtained from the more sophisticated weight function (5.2) used for Fig. 4(a). Note, however, that this weight function does not account for the amplitude perturbations. It follows from
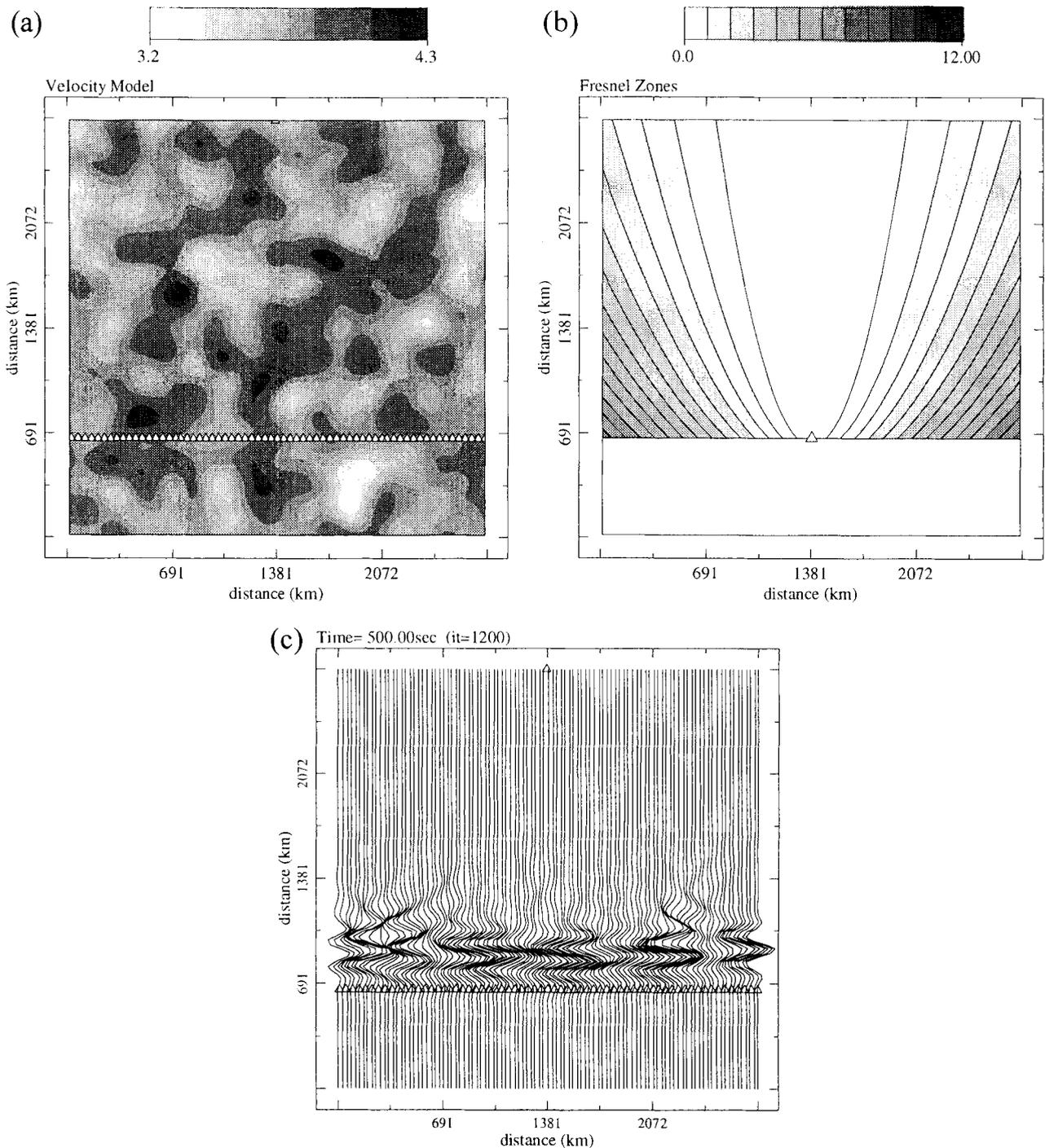
Figure 3. (a) The Gaussian quasi-random velocity model used for the numerical experiment of Section 5. A plane wave is incident from above, and the wavefield is sampled at receivers indicated by triangles. (b) Contour diagram of the detour time D/v for the receiver indicated by the triangle. The first contour level corresponds to a detour time of half a period and is hence an indication of the location of the first Fresnel zone. (c) The wavefield recorded at the receivers indicated by triangles in (a).

expression (3.15) that a weight function with a non-zero imaginary component is needed for this.

In Fig. 6, a finite-difference simulation of the wavefield that propagates through the realization of an exponential random medium is shown. Note that the gross features of the velocity perturbations in Fig. 6(a) are similar to the velocity perturbations in the realization of the Gaussian random medium in Fig. 3(a). The main difference is that the exponential quasi-

random medium is much richer in short-wavelength structures—this is due to the fact that its power spectrum decays algebraically with wavenumber rather than exponentially (Frankel & Clayton 1986). The exponential quasi-random medium thus imposes a stronger test on the theory presented here.

The traveltime and amplitude changes computed for the exponential quasi-random medium of Fig. 6(a) using the same
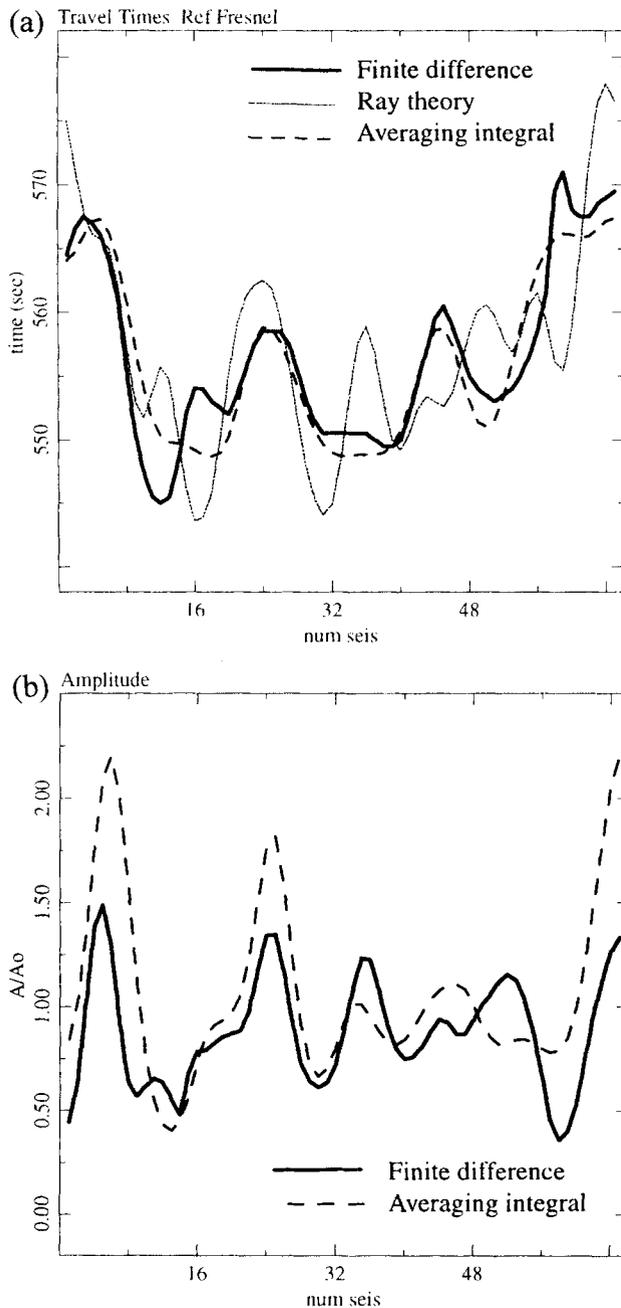
Figure 4. (a) Arrival-time perturbation for the wavefield recorded at the triangles in Fig. 3(a) computed from the averaging integral (3.14) using the replacement (4.6) with the weight function W given by (5.2). (b) Amplitude perturbation for the wavefield recorded at the triangles in Fig. 3(a) computed from the averaging integral (3.15) using the replacement (4.6) with the weight function W given by (5.2).

Figure 5. (a) Arrival-time perturbation for the wavefield recorded at the triangles in Fig. 3(a) computed from the averaging integral (3.14) using the replacement (4.6). The weight function W is equal to unity for $kD \leq \pi/2$ and is equal to zero for larger values of the detour. (b) Amplitude perturbation for the wavefield recorded at the triangles in Fig. 3(a) computed from the averaging integral (3.15) using the replacement (4.6). The weight function W is equal to unity for $kD \leq \pi/2$ and is equal to zero for larger values of the detour.

method as used for Figs 4(a) and (b) are shown in Figs 7(a) and (b). A comparison with Figs 4(a) and (b) shows that the traveltime and amplitude anomalies obtained from the finite-difference experiments and the averaging integrals for both the Gaussian and the exponential quasi-random medium are very similar, despite the fact that the exponential quasi-random medium contains much more short-wavelength structure. This is due to two effects. First, the averaging properties of the wavefield effectively smooth the medium through the averaging

integrals (3.14) and (3.15). Second, the traveltime perturbations are dominated by first-order effects in these examples. It is known that in second order, the Gaussian and exponential quasi-random media have dramatically different effects on the traveltime (Roth *et al.* 1993; Witte, Roth & Müller 1996). Since the theory of this paper only accounts for the first-order effects of slowness perturbations on the traveltime, the propagation

(a)

Velocity Model

(b)

Fresnel Zones

(c)   Time= 500.00sec (it=1200)
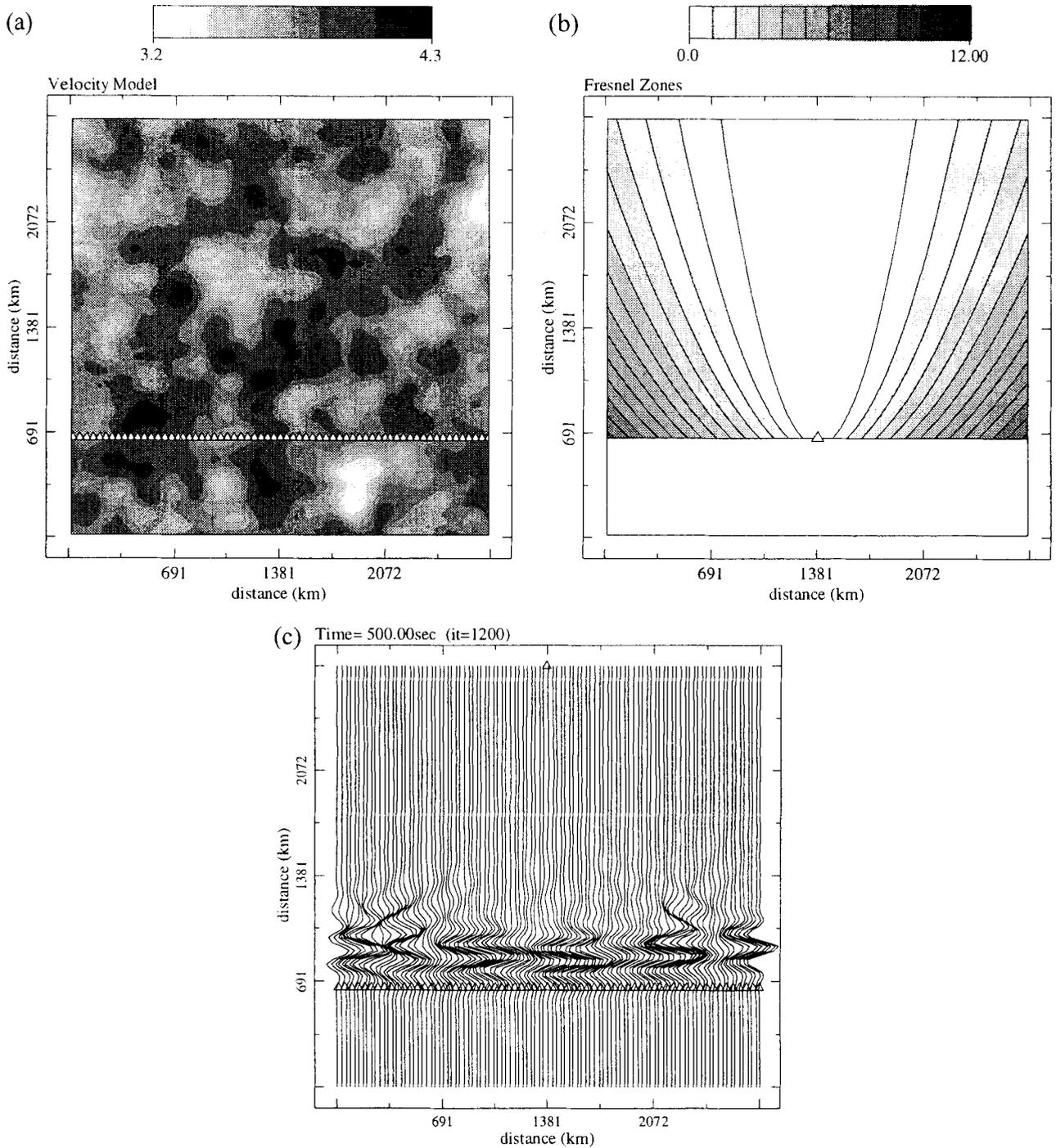
Figure 6. (a) As Fig. 3(a) but for the exponential quasi-random medium. (b) As Fig. 3(b) but for the exponential quasi-random medium. (c) As Fig. 3(c) but for the exponential quasi-random medium.

distance in the examples of Figs 3 and 6 is chosen in such a way that ray bending, and the associated second-order effect on the traveltime, is small.

6 GENERAL SMOOTH REFERENCE MEDIUM IN THREE DIMENSIONS

The averaging integral derived in Section 3 is for the special case of a homogeneous reference medium. However, the theory

can be generalized for inhomogeneous reference media $v(\mathbf{r})$ that are sufficiently smooth so that ray theory can be used to describe the properties of the unperturbed wave $u_0(\mathbf{r})$ and the Green's function $G(\mathbf{r}_0, \mathbf{r})$ of the reference medium. The derivation of the averaging integral is shown in this section for the 3-D case. The analysis is based on the dynamic ray theory of Červený & Hron (1980), hereafter referred to as CH. Note that, in contrast to the work of CH, the density is assumed to be constant in this paper.

(a) Travel Times Ref Fresnel

Finite difference — Ray theory — — — Averaging integral

(b) Amplitude

Finite difference — — — Averaging integral
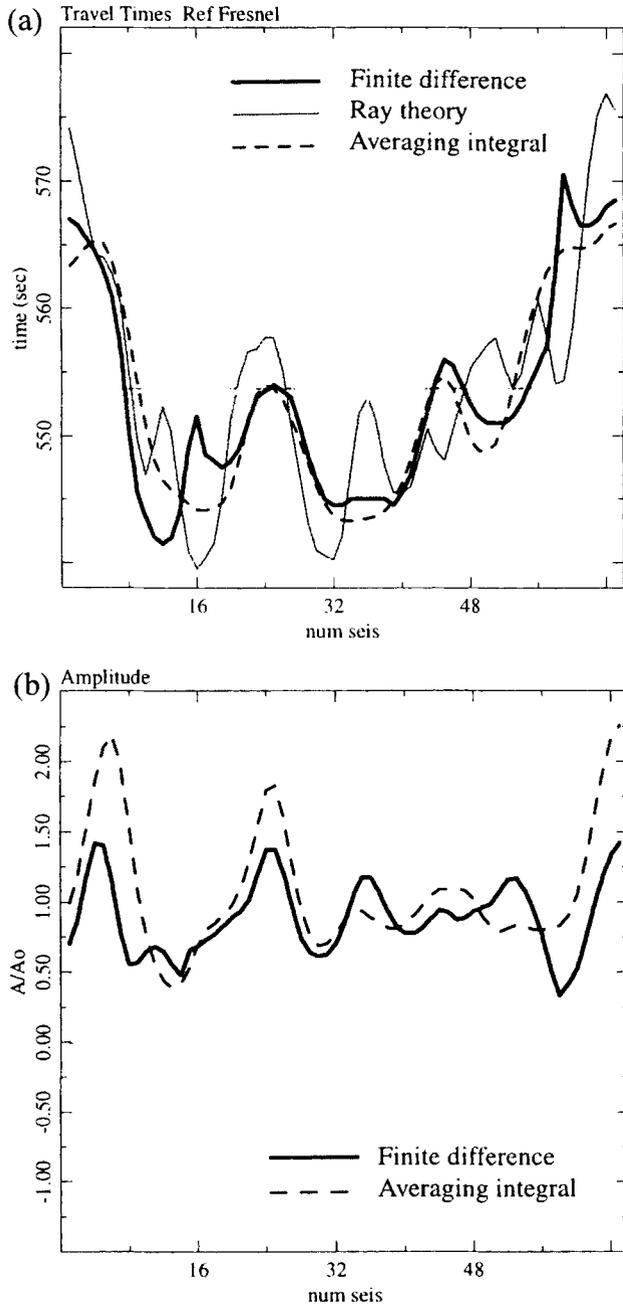
Figure 7. (a) As Fig. 4(a) but for the exponential quasi-random medium. (b) As Fig. 4(b) but for the exponential quasi-random medium.

According to eq. (39) of CH, the amplitude A of a ray-geometrical solution satisfies

$$A(s_1) = A(s_2)\sqrt{\frac{J(s_1)v(s_1)}{J(s_2)v(s_2)}}. \qquad (6.1)$$

In this expression, s_1 and s_2 denote arclengths along the same ray and $J(s)$ is the geometrical spreading. This result implies that the Green's function satisfies

$$G(\mathbf{r}_1, \mathbf{r}_2) = C\frac{\exp(i\omega\tau(\mathbf{r}_1, \mathbf{r}_2))}{\sqrt{J(\mathbf{r}_1, \mathbf{r}_2)v(\mathbf{r}_1)}}, \qquad (6.2)$$

where $\tau(\mathbf{r}_1, \mathbf{r}_2)$ is the traveltime between $\mathbf{r}_2$ and $\mathbf{r}_1$, while $J(\mathbf{r}_1, \mathbf{r}_2)$ is the geometrical spreading in $\mathbf{r}_1$ due to a point source in $\mathbf{r}_2$. The source parameter C follows from the requirement that, when $\mathbf{r}_1$ and $\mathbf{r}_2$ are separated by a small distance $\eta = |\mathbf{r}_1 - \mathbf{r}_2|$, the Green's function is the same as if the medium was locally homogeneous. This implies that

$$G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{4\pi}\frac{\exp(ik|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} = -\frac{1}{4\pi}\frac{\exp(i\omega\eta/v(\mathbf{r}_2))}{\eta}. \qquad (6.3)$$

As shown by CH, the geometrical spreading for this case is given by

$$J(\mathbf{r}_1, \mathbf{r}_2) = |\mathbf{r}_1 - \mathbf{r}_2|^2 = \eta^2, \qquad (6.4)$$

and the traveltime is given by $\tau(\mathbf{r}_1, \mathbf{r}_2) = \eta/v(\mathbf{r}_2)$. Comparing these results with (6.2) allows for the determination of the source parameter C:

$$C = -\frac{1}{4\pi}\sqrt{v(\mathbf{r}_2)}, \qquad (6.5)$$

and hence the Green's function is given by

$$G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{4\pi}\sqrt{\frac{v(\mathbf{r}_2)}{v(\mathbf{r}_1)}}\frac{\exp(i\omega\tau(\mathbf{r}_1, \mathbf{r}_2))}{\sqrt{J(\mathbf{r}_1, \mathbf{r}_2)}}. \qquad (6.6)$$

Similarly, the incoming wave is given by

$$u_0(\mathbf{r}) = A(\mathbf{r})\exp(i\omega\tau(\mathbf{r})). \qquad (6.7)$$

In the remainder of this paper, $A(\mathbf{r})$, $\tau(\mathbf{r})$ and $J(\mathbf{r})$ denote the amplitude, arrival time and geometrical spreading of the direct wave, while $A(\mathbf{r}_1, \mathbf{r}_2)$, $\tau(\mathbf{r}_1, \mathbf{r}_2)$ and $J(\mathbf{r}_1, \mathbf{r}_2)$ denote the corresponding quantities at location $\mathbf{r}_1$ for a point source in $\mathbf{r}_2$.

The relations (6.6) and (6.7) can be used in the scattering integral (2.5). In doing so, the traveltime $\tau(\mathbf{r}_0, \mathbf{r})$ and the geometrical spreading $J(\mathbf{r}_0, \mathbf{r})$ are by virtue of the principle of reciprocity replaced by $\tau(\mathbf{r}, \mathbf{r}_0)$ and $J(\mathbf{r}, \mathbf{r}_0)$ respectively. This gives

$$u_B(\mathbf{r}_0) = \frac{1}{4\pi}\int\frac{\omega^2}{v^2(\mathbf{r})}\sqrt{\frac{v(\mathbf{r})}{v(\mathbf{r}_0)}}\frac{\exp(i\omega\tau(\mathbf{r}, \mathbf{r}_0))}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}}$$
$$\times n(\mathbf{r})A(\mathbf{r})\exp(i\omega\tau(\mathbf{r}))\,dV. \qquad (6.8)$$

By analogy with (3.5), define the delay time $T(\mathbf{r}_0, \mathbf{r})$ by

$$T(\mathbf{r}_0, \mathbf{r}) \equiv \tau(\mathbf{r}) + \tau(\mathbf{r}, \mathbf{r}_0) - \tau(\mathbf{r}_0). \qquad (6.9)$$

In this case, the delay time is used because it is this quantity that describes the relative timing of scattered wave arrivals. For the special case of a homogeneous medium, this quantity is proportional to the detour D.

In order to carry out an analysis similar to that in Section 3, it is advantageous to convert the volume integral in (6.8) to ray-centred coordinates. Ray-centred coordinates (s, q_1, q_2) are defined using a ray in the reference medium $v(\mathbf{r})$ that arrives at the observation point $\mathbf{r}_0$. The first coordinate s denotes the arclength along this reference ray. The coordinates q_1 and q_2 denote two coordinates perpendicular to the reference ray. The reader is referred to CH for details. Using eqs (42), (43) and (55) of CH, one obtains the following expression for a volume element dV in ray-centred coordinates:

$$dV = h(s, q_1, q_2)\,ds\,dq_1\,dq_2, \qquad (6.10)$$

with

$$h(s, q_1, q_2) = 1 + \frac{1}{v}(\mathbf{q} \cdot \nabla v), \tag{6.11}$$

where $\mathbf{q}$ is the vector perpendicular to the reference ray with components q_1 and q_2 in ray-centred coordinates. Converting the volume integral in (6.8) with these results, using (6.6) and (6.9) and multiplying and dividing (6.8) by $A(\mathbf{r}_0)$ one obtains, using (6.7),

$$u_B(\mathbf{r}_0) = \frac{\omega^2}{4\pi} u_0(\mathbf{r}_0) \int \left\{ \iiint \frac{A(\mathbf{r})}{A(\mathbf{r}_0)} \sqrt{\frac{v(\mathbf{r})}{v(\mathbf{r}_0)}} \frac{n(\mathbf{r})}{v^2(\mathbf{r})} \frac{\exp(i\omega T(\mathbf{r}, \mathbf{r}_0))}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}} \right.$$

$$\left. \times h(s, q_1, q_2) \, dq_1 dq_2 \right\} ds. \tag{6.12}$$

As a next step, consider the integral

$$I(s) \equiv \iint_{-\infty}^{\infty} \frac{A(\mathbf{r})}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}} \exp(i\omega T(\mathbf{r}, \mathbf{r}_0)) h(s, q_1, q_2) \, dq_1 dq_2. \tag{6.13}$$

Analogous to eq. (3.9), this integral is evaluated using the stationary-phase approximation. Let the matrix $\mathbf{M}$ be defined as in expression (50) of CH: $M_{ij} \equiv \partial^2 \tau / \partial q_i \partial q_j$. The reference ray is a curve along which the traveltime is stationary. It is thus sufficient to prescribe the matrix $\mathbf{M}$ in order to know how the traveltime changes when one moves away from the reference ray. Let this matrix be denoted by $\mathbf{M}^{\text{in}}$ for the incident wave. The traveltime of the incident wave to location (s, q_1, q_2) is given by

$$\tau(s, q_1, q_2) = \tau(s) + \frac{1}{2} \mathbf{q}^{\mathsf{T}} \mathbf{M}^{\text{in}} \mathbf{q}. \tag{6.14}$$

In this expression and following expressions, $f(s)$ denotes a quantity evaluated on the reference ray:

$$f(s) \equiv f(s, q_1 = 0, q_2 = 0). \tag{6.15}$$

In a similar way, let the matrix $\mathbf{M}^{\text{out}}$ be the second-derivative matrix of the traveltime $\tau(\mathbf{r}, \mathbf{r}_0)$, which is equivalent to the traveltime from a fictitious point source at $\mathbf{r}_0$ to location $\mathbf{r}$ (see Fig. 6):

$$\tau(\mathbf{r}, \mathbf{r}_0) = \tau(s, s_0) + \frac{1}{2} \mathbf{q}^{\mathsf{T}} \mathbf{M}^{\text{out}} \mathbf{q}, \tag{6.16}$$

where $\tau(s, s_0)$ is the traveltime along the reference ray from the point $(s_0, 0, 0)$ to $(s, 0, 0)$ (see Fig. 8). Using the identity $\tau(s_0) = \tau(s) + \tau(s, s_0)$ and using the definition (6.9) one finds that

$$T(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2} \mathbf{q}^{\mathsf{T}} (\mathbf{M}^{\text{in}} + \mathbf{M}^{\text{out}}) \mathbf{q}. \tag{6.17}$$

This result can be used in the stationary-phase analysis of the integral (6.13). The $(\mathbf{q} \cdot \nabla v)/v$ term that is contained within the scale factor $h(s, q_1, q_2)$ gives a vanishing contribution to the stationary-phase approximation of the integral (6.13) because it leads to an integral over q from $-\infty$ to ∞ of an integrand that is an odd function of q. Performing a multidimensional stationary-phase analysis (Bleistein 1984) of the integral (6.13) gives

$$I(s) \equiv \iint_{-\infty}^{\infty} \frac{A(\mathbf{r})}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}} \exp\left(\frac{i}{2} \omega \mathbf{q}^{\mathsf{T}} (\mathbf{M}^{\text{in}} + \mathbf{M}^{\text{out}}) \mathbf{q}\right) dq_1 dq_2$$

$$= \frac{2\pi}{\omega} \frac{\exp\left(\frac{i\pi}{4} \text{sgn}(\mathbf{M}^{\text{in}} + \mathbf{M}^{\text{out}})\right)}{\sqrt{|\det(\mathbf{M}^{\text{in}} + \mathbf{M}^{\text{out}})|}} \frac{A(s)}{\sqrt{J(s, s_0)}}, \tag{6.18}$$

where sgn $\mathbf{M}$ is the number of positive eigenvalues of $\mathbf{M}$ minus the number of negative eigenvalues of $\mathbf{M}$.

The square root of the determinant of a 2×2 matrix $\mathbf{M}$ has to be defined with some care. Let $\mathbf{M}$ have eigenvalues λ_1 and λ_2, then $\det \mathbf{M} = \lambda_1 \lambda_2$. In (6.18), the square root is taken from the absolute value of the determinant—this quantity is well defined. In general, we define the square root of a determinant to be

$$\sqrt{\det \mathbf{M}} \equiv i^{\text{number of negative eigenvalues}} \sqrt{|\det \mathbf{M}|}. \tag{6.19}$$

For a 2×2 matrix, this is equivalent to

$$\sqrt{\det \mathbf{M}} = i \exp\left(-i\frac{\pi}{4} \text{sgn}(\mathbf{M})\right) \sqrt{|\det \mathbf{M}|}. \tag{6.20}$$

Using this result, the stationary-phase integral (6.18) is given by

$$I(s) = \frac{2\pi i}{\omega} \frac{A(s)}{\sqrt{J(s, s_0)}} \bigg/ \sqrt{\det(\mathbf{M}^{\text{in}} + \mathbf{M}^{\text{out}})}. \tag{6.21}$$

It is convenient to use the curvature matrix $\mathbf{K} = v\mathbf{M}$ defined in
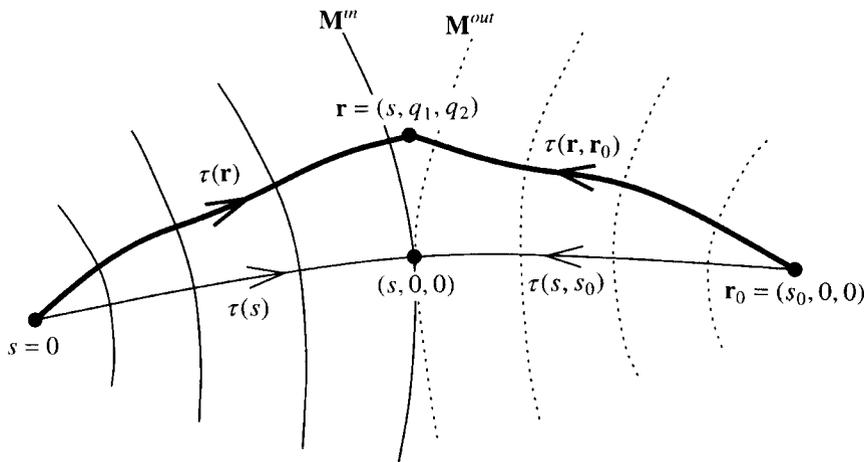
Figure 8. Definition of the geometric variables for the case of an inhomogeneous reference velocity.

eq. (68) of CH rather than the matrix **M**. This gives

$$I(s) = \frac{2\pi i v(s)}{\omega} \frac{A(s)}{\sqrt{J(s, s_0)}} \bigg/ \sqrt{\det(\mathbf{K}^{\text{in}} + \mathbf{K}^{\text{out}})}. \tag{6.22}$$

Because the integrand is evaluated in the stationary-phase approximation on the reference ray, the curvature matrices and the velocity are evaluated on the reference ray. An averaging integral analogous to (3.10) is obtained by multiplying and dividing the q-integrals between curly brackets in (6.12) by $I(s)$. Using (6.22) for the numerator and (6.13) for the denominator, this gives

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2v(\mathbf{r}_0)} u_0(\mathbf{r}_0) \int \left\{ \int \int \left(\frac{A(s)}{\sqrt{J(s, s_0)}} \frac{A(\mathbf{r})}{A(\mathbf{r}_0)} \sqrt{\frac{v(\mathbf{r})}{v(\mathbf{r}_0)}} v(\mathbf{r}_0) v(s) \right. \right.$$

$$\left. \times \frac{n(\mathbf{r})}{v^2(\mathbf{r})} \frac{e^{i\omega T(\mathbf{r}, \mathbf{r}_0)}}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}} \frac{h(s, q_1, q_2)}{\sqrt{\det(\mathbf{K}^{\text{in}} + \mathbf{K}^{\text{out}})}} \right) dq_1 dq_2 \bigg/$$

$$\left. \int \int \left(\frac{A(\mathbf{r})}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}} e^{i\omega T(\mathbf{r}, \mathbf{r}_0)} \right) dq_1 dq_2 \right\} ds. \tag{6.23}$$

In deriving this expression, a term $v(\mathbf{r}_0)$ has been factored out in front.

This expression constitutes an averaging integral that can be used with eqs (2.7) and (2.8) for obtaining the perturbations of the phase and the amplitude. For monochromatic signals, the integral (6.23) can immediately be implemented. Similarly to the averaging integral (3.10) for a homogeneous reference medium, one averages not simply the perturbation $n(\mathbf{r})$, but a more complicated expression that contains amongst other things the amplitude of the incident wave and the geometrical spreading. In addition, the averaging integral also contains the local wavefront curvature (through the curvature matrices $\mathbf{K}^{\text{in}}$ and $\mathbf{K}^{\text{out}}$) as well as the velocity variations of the reference medium. The latter effect is due to the fact that the perturbation of the Helmholtz equation (2.1) is given by $n(\mathbf{r})/v^2(\mathbf{r})$, and hence $n(\mathbf{r})$ is weighted by the velocity.

As a consistency check with the results of Section 3, consider the averaging integral (6.23) for the special case of a homogeneous reference medium: $v(\mathbf{r}) = v = \text{const}$. Consider a point source at the origin; in that case, referring to Fig. 1, one has $A(\mathbf{r})/A(\mathbf{r}_0) = z_0/|\mathbf{r}|$, and $J(\mathbf{r}, \mathbf{r}_0) = |\mathbf{r} - \mathbf{r}_0|^2$. The curvature matrices $\mathbf{K}^{\text{in}}$ and $\mathbf{K}^{\text{out}}$ are diagonal matrices with diagonal elements $1/z$ and $1/(z_0 - z)$ respectively, hence

$$\det(\mathbf{K}^{\text{in}} + \mathbf{K}^{\text{out}}) = \left(\frac{1}{z} + \frac{1}{z_0 - z} \right)^2 = \left[\frac{z_0}{z(z_0 - z)} \right]^2. \tag{6.24}$$

Furthermore $A(s)/\sqrt{J(s, s_0)} = 1/z(z_0 - z)$. For straight reference rays, the scale factor h equals unity. Use of these results gives

$$\frac{A(s)}{\sqrt{J(s, s_0)}} \frac{A(\mathbf{r})}{A(\mathbf{r}_0)} \sqrt{\frac{v(\mathbf{r})}{v(\mathbf{r}_0)}} v(\mathbf{r}_0) v(s) \frac{h(s, q_1, q_2)}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)} \sqrt{\det(\mathbf{K}^{\text{in}} + \mathbf{K}^{\text{out}})}}$$

$$= \frac{1}{|\mathbf{r}| |\mathbf{r}_0 - \mathbf{r}|}, \tag{6.25}$$

and

$$\frac{A(\mathbf{r})}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}} = \frac{1}{|\mathbf{r}| |\mathbf{r}_0 - \mathbf{r}|}. \tag{6.26}$$

This implies that for this case both weight functions in the

numerator and denominator of (6.23) are equal to the amplitude factor $F(\mathbf{r}_0, \mathbf{r})$ for a point source in three dimensions given by expression (3.13). It thus follows that the averaging integral (6.23) reduces to the averaging integral (3.10) obtained earlier for the special case of a point source in a homogeneous reference medium in three dimensions.

7 THE AVERAGING INTEGRAL IN THREE DIMENSIONS FOR TRANSIENT SIGNALS

In this section the effect of the perturbation $n(\mathbf{r})$ on transient signals is considered. As argued in Section 4, in this case one needs to consider the contribution in the averaging integral (6.23) from the first Fresnel zone. It is assumed that the reference medium is sufficiently smooth to warrant the use of ray theory for the incident wave and the Green's function for the reference medium. This implies that it is assumed that both $v(\mathbf{r})$ and $A(\mathbf{r})$ do not vary appreciably over the first Fresnel zone. It is consistent with this assumption to make the following replacements when one only considers the contributions from the first Fresnel zone to the averaging integral (6.23): $A(\mathbf{r}) \to A(s)$, $v(\mathbf{r}) \to v(s)$, $J(\mathbf{r}, \mathbf{r}_0) \to J(s, s_0)$, etc. This implies that whenever these quantities are evaluated within the first Fresnel zone they can be replaced by the corresponding value on the reference ray. Since the observation point $\mathbf{r}_0$ is located on the reference ray, one can obviously carry out the substitutions for $\mathbf{r}_0$. Using (6.1) one finds that the averaging integral (6.23) reduces to

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2v(\mathbf{r}_0)} u_0(\mathbf{r}_0) \int \left\{ \int \int \left(\frac{v(s_0)}{v(s)} \frac{\sqrt{J(s_0)}}{\sqrt{J(s)}\sqrt{J(s, s_0)}} n(\mathbf{r}) \right. \right.$$

$$\left. \times \frac{e^{i\omega T(\mathbf{r}, \mathbf{r}_0)}}{\sqrt{\det(\mathbf{K}^{\text{in}} + \mathbf{K}^{\text{out}})}} h(s, q_1, q_2) \right) dq_1 dq_2 \bigg/$$

$$\left. \int \int (e^{i\omega T(\mathbf{r}, \mathbf{r}_0)}) dq_1 dq_2 \right\} ds. \tag{7.1}$$

This expression can be simplified further by using the following identity:

$$J(s)J(s, s_0) \det(\mathbf{K}^{\text{in}} + \mathbf{K}^{\text{out}}) = J(s_0)v^2(s)/v^2(s_0) \tag{7.2}$$

(see Appendix D for the derivation). Inserting this relation into the averaging integral (7.1) gives

$$u_B(\mathbf{r}_0) = \frac{i\omega}{2v(\mathbf{r}_0)} u_0(\mathbf{r}_0) \int \frac{\int \int \frac{v^2(s_0)}{v^2(s)} n(\mathbf{r}) e^{i\omega T(\mathbf{r}, \mathbf{r}_0)} h(s, q_1, q_2) dq_1 dq_2}{\int \int e^{i\omega T(\mathbf{r}, \mathbf{r}_0)} dq_1 dq_2} ds. \tag{7.3}$$

This averaging integral expresses by virtue of the relations (2.7) and (2.8) the phase and amplitude perturbations of transient arrivals as weighted averages of $n(\mathbf{r})$ over the first Fresnel zone. the dimensionless weight factor $v^2(s_0)/v^2(s)$ is present because the perturbation of the Helmholtz equation (2.1) is $n(\mathbf{r})/v^2(\mathbf{r})$ rather than $n(\mathbf{r})$. The geometric term $h(s, q_1, q_2)$ accounts for the fact that the ray-centred coordinates are not Cartesian.

Expression (7.3) forms a starting point for further approximations. In practical implementations, a substitution similar

to (4.6) can be useful. Finally, note that the weight function in the averaging integral (7.3) does not depend on whether one has passed any caustics. At caustics, the geometrical spreading vanishes, and after caustics, phase shifts that are multiples of $\exp(i\pi/2)$ can occur (Chapman & Drummond 1982; Choy & Richards 1975). These phase shifts are implicitly present in expression (7.1) because the square roots of the geometrical spreading are taken. However, the steps leading to the averaging integral (7.3) show that the factors $\exp(i\pi/2)$ due to caustics cancel in the final result. This reflects the fact that a slow (fast) anomaly within the first Fresnel zone causes a later (earlier) arrival, regardless of whether that slow anomaly is sampled by a wave that has travelled through a caustic or not.

8 MODIFICATIONS FOR THE 2-D CASE

The theory of Sections 6 and 7 can be reformulated for the case of two spatial dimensions. The derivation is simpler than for the 3-D case because the curvature matrix is a 1×1 matrix; the determinant, the trace and the matrix itself are identical. However, there are some subtle differences in the derivation. In this section, the modification to the theory of Sections 6 and 7 is shown in order to drive the corresponding results for the case of two dimensions.

The general form of the Green's function is given by (6.2). The source parameter C follows by comparing this expression with eq. (3.3) when the point r_1 is located at an infinitesimal distance η from r_2. This gives, for the ray-geometrical Green's function in two dimensions,

$$G(r_1, r_2) = \frac{-e^{i\pi/4}}{\sqrt{8\pi\omega/v(r_2)}} \sqrt{\frac{v(r_2)}{v(r_1)}} \frac{e^{i\omega\tau(r_1, r_2)}}{\sqrt{J(r_1, r_2)}}.$$ (8.1)

The vector q of the transverse coordinate of the ray-centred coordinates is replaced by a single component q, and the Jacobian of the ray-centred coordinates is given by

$$h(s, q) = 1 + \frac{1}{v(s)} q\partial_q v(s, q = 0),$$ (8.2)

where ∂_q is the derivative perpendicular to the reference ray. Using these results one arrives, instead of at (6.12), at the following expression for the Born field:

$$u_B(r_0) = \omega \sqrt{\frac{\omega}{8\pi}} e^{i\pi/4} u_0(r_0)$$

$$\times \int \left\{ \int\int \frac{A(r)}{A(r_0)} \frac{1}{v(r)\sqrt{v(r_0)}} n(r) \frac{e^{i\omega T(r, r_0)}}{\sqrt{J(r, r_0)}} h(s, q) \, dq \right\} ds.$$ (8.3)

The integral $I(s)$ can be defined analogously to (6.13). The matrices M and K are replaced by scalars, hence the trace and the determinant are equal. By analogy to (6.18) this gives

$$I(s) = \int_{-\infty}^{\infty} e^{i\omega T(r, r_0)} \, dq = \int_{-\infty}^{\infty} \exp(i\omega(M^{in} + M^{out})q^2) \, dq$$

$$= \sqrt{\frac{2\pi}{\omega}} \frac{e^{i\pi/4}}{\sqrt{M^{in} + M^{out}}},$$ (8.4)

where the convention $\sqrt{M} = +i\sqrt{|M|}$ is used when $M < 0$. Taking the same steps as in Section 7, one obtains the

equivalent of expression (6.23):

$$u_B(r_0) = \frac{i\omega}{2v(r_0)} u_0(r_0)$$

$$\times \int \frac{\int \frac{A(r)}{A(r_0)} \frac{\sqrt{v(r_0)v(s)}}{v(r)} n(r) \frac{e^{i\omega T(r, r_0)}}{\sqrt{J(r, r_0)}} \frac{h(s, q)}{\sqrt{K^{in} + K^{out}}} \, dq}{\int e^{i\omega T(r, r_0)} \, dq} \, ds.$$

(8.5)

This expression can be used to compute the first-order perturbation of the phase and amplitude of monochromatic signals using (2.7) and (2.8).

For transient signals, an analysis similar to the one shown in Section 7 can be carried out. The equivalent expression of (7.1) for the 2-D case is given by

$$u_B(r_0) = \frac{i\omega}{2v(r_0)} u_0(r_0)$$

$$\times \int \frac{\int \left(\frac{v(s_0)}{v(s)}\right)^{3/2} \frac{\sqrt{J(s_0)}}{\sqrt{J(s)}\sqrt{J(s, s_0)}} n(r) \frac{e^{i\omega T(r, r_0)}}{\sqrt{K^{in} + K^{out}}} h(s, q) \, dq}{\int e^{i\omega T(r, r_0)} \, dq} \, ds.$$

(8.6)

Note that the main difference with (7.1) for the 3-D case is the weight factor $[v(s_0)/v(s)]^{3/2}$ in (8.6) rather than $v(s_0)/v(s)$ in (7.1). The integral can be simplified further by using the following identity:

$$J(s)J(s, s_0)(K^{in} + K^{out}) = J(s_0)v(s)/v(s_0)$$ (8.7)

(see Appendix E for the derivation). Note the change in the power of $v(s)/v(s_0)$ compared with (7.2) for the 3-D case. Inserting this result in (8.6) gives

$$u_B(r_0) = \frac{i\omega}{2v(r_0)} u_0(r_0) \int \frac{\int \frac{v^2(s_0)}{v^2(s)} n(r) e^{i\omega T(r, r_0)} h(s, q) \, dq}{\int e^{i\omega T(r, r_0)} \, dq} \, ds.$$ (8.8)

This averaging integral, including the weight factor $v^2(s_0)/v^2(s)$, has exactly the same functional form as (7.3) for the 3-D case. This must be the case, because both in two and in three dimensions the phase shift of transient arrivals depends on the average of $n(r)/v^2(r)$ over the first Fresnel zone. Note that in contrast to this result, the corresponding integrals (7.1) and (8.6) for three and two dimensions respectively appear to have different integrands.

9 DISCUSSION

The averaging integrals derived here imply that, to first order, the phase shift and amplitude perturbations of transient wave arrivals due to a velocity perturbation are given by a weighted average of the velocity perturbation over the first Fresnel zone. This explains why tomographic reconstruction methods based on ray theory can be used even for media where perturbations are present on short length-scales that violate the requirements for the use of ray theory. The theory of this paper implies that the delay time is given by a weighted average of the velocity perturbation over the first Fresnel zone. When one uses

ray theory for tomographic reconstructions, one effectively collapses the true weight function over the first Fresnel zone to a line integral along a geometric ray (the centre of the first Fresnel zone). Given the fact that the resolution in practical tomographic inversions is finite and that one often regularizes the inverse problem with a smoothness constraint, it is not surprising that collapsing the true weight function to a line does not significantly alter the reconstructed images since the reconstructed images are in general blurred versions of the true medium anyhow.

One should, however, be careful in over-interpreting this result. The theory presented here only accounts for the first-order changes in the phase and amplitude of the wavefield due to velocity perturbations. This implies that ray bending effects and true multiple scattering phenomena are not accounted for by the present theory; these effects will lead to higher-order changes to the phase and amplitude. This implies that the conclusion that the phase and amplitude perturbations can to first order be expressed as weighted averages of the velocity perturbation over the first Fresnel zone is only useful in situations where these higher-order effects are of minor importance. Fortunately, in many applications such as solid earth tomography, the heterogeneity is indeed only of the order of a few per cent (Gudmundsson, Davies & Clayton 1990). It is thus the weakness of the heterogeneity that justifies the use of ray-geometric tomographic inversions in applications such as solid earth tomography.

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APPENDIX A: DERIVATION OF THE RYTOV APPROXIMATION

The Rytov approximation is obtained by writing the wavefield as

$$u(\mathbf{r}) = \exp(S(\mathbf{r})), \tag{A1}$$

and deriving the perturbation of $S(\mathbf{r})$. In this way, one can derive the first-order perturbation of the phase of the wavefield. This is achieved by inserting the transformation (A1) in (2.1), which gives

$$\nabla^2 S(\mathbf{r}) + |\nabla S(\mathbf{r})|^2 + \frac{\omega^2}{v^2(\mathbf{r})} = -\frac{\omega^2}{v^2(\mathbf{r})} n(\mathbf{r}), \tag{A2}$$

and by inserting the perturbation series

$$S(\mathbf{r}) = S_0(\mathbf{r}) + S_1(\mathbf{r}) + \cdots, \tag{A3}$$

where S_j depends on the perturbation $n(\mathbf{r})$ in jth order. The contributions of zeroth and first orders in $n(\mathbf{r})$ lead to the following expressions:

$$\nabla^2 S_0(\mathbf{r}) + |\nabla S_0(\mathbf{r})|^2 + \frac{\omega^2}{v^2(\mathbf{r})} = 0, \tag{A4}$$

$$\nabla^2 S_1(\mathbf{r}) + 2(\nabla S_0 \cdot \nabla S_1) = -\frac{\omega^2}{v^2(\mathbf{r})} n(\mathbf{r}). \tag{A5}$$

Eq. (A4) is equivalent to the Helmholtz equation for the unperturbed problem, hence

$$u_0(\mathbf{r}) = \exp(S_0(\mathbf{r})). \tag{A6}$$

Eq. (A5) can be solved using the following substitution:

$$S_1(\mathbf{r}) = f(\mathbf{r}) \exp(-S_0(\mathbf{r})). \tag{A7}$$

This gives, using (A6),

$$\nabla^2 f(\mathbf{r}) + \frac{\omega^2}{v^2(\mathbf{r})} f(\mathbf{r}) = -\frac{\omega^2}{v^2(\mathbf{r})} n(\mathbf{r}) u_0(\mathbf{r}). \tag{A8}$$

Using the unperturbed Green's function $G(\mathbf{r}_0, \mathbf{r})$, this equation can be solved to give

$$f(\mathbf{r}_0) = -\int \frac{\omega^2}{v^2(\mathbf{r})} G(\mathbf{r}_0, \mathbf{r}) n(\mathbf{r}) u_0(\mathbf{r}) \, dV = u_B(\mathbf{r}_0), \tag{A9}$$

where (2.5) has been used in the last identity.

The Rytov approximation is obtained by taking only the zeroth- and first-order terms in the series (A3) into account:

$$u_R(\mathbf{r}) \equiv \exp(S_0(\mathbf{r}) + S_1(\mathbf{r})). \tag{A10}$$

Using the expressions (A6), (A7) and (A9), one obtains relation (2.6) between the Rytov field and the Born field.

APPENDIX B: CORRESPONDENCE WITH RAY-GEOMETRICAL RESULTS

In order to establish the correspondence with ray theory, consider the special case of a perturbation that is smooth both on the scale of a wavelength and on the scale of the width of the first Fresnel zone. In that case the requirements for ray theory are satisfied, and the expressions (3.14) and (3.15) should lead to the correct ray-geometrical results. This is explicitly verified in this section. Under these smoothness conditions, the integrals in the numerator of (3.14) and (3.15) can be solved in the stationary-phase approximation; this entails a second-order Taylor expansion of the perturbation:

$$n(x, z) = n(0, z) + x\partial_x n(x = 0, z) + \frac{1}{2} x^2 \partial_{xx} n(x = 0, z). \tag{B1}$$

Consider the special case of a plane incoming wave in two dimensions: the amplitude factor F in expression (3.7) has, to second order in x, the following expansion:

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{1}{\sqrt{|z_0 - z|}} \left(1 - \frac{x^2}{2(z_0 - z)^2}\right). \tag{B2}$$

The last term arises from the variation of the geometrical spreading with the transverse distance x. The expansions (B1), (B2) and (3.8) can be used for the stationary-phase evaluation of the integrals in (3.10). The term proportional to x in (B1) does not contribute. Taking terms up to order x^2 into account

gives

$$\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) e^{ik_0 D(\mathbf{r})} \, dx$$

$$= \sqrt{\frac{2\pi}{k_0}} e^{i\pi/4} \left\{ n + \frac{i}{2} \partial_{xx} n \frac{|z_0 - z|}{k_0} - \frac{i}{4} \frac{n}{k_0|z_0 - z|} \right\}. \tag{B3}$$

Setting $n = 1$ gives

$$\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) e^{ik_0 D(\mathbf{r})} \, dx = \sqrt{\frac{2\pi}{k_0}} e^{i\pi/4} \left\{ 1 - \frac{i}{4} \frac{1}{k_0|z_0 - z|} \right\}. \tag{B4}$$

Note that the term $1/k_0|z_0 - z|$ is due to the variation of the amplitude factor F with the transverse distance. Using these results one finds that in the far field ($k_0|z_0 - z| \gg 1$)

$$\frac{\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) n(\mathbf{r}) e^{ik_0 D(\mathbf{r})} \, dx}{\int_{-\infty}^{\infty} F(\mathbf{r}, \mathbf{r}_0) e^{ik_0 D(\mathbf{r})} \, dx} = n(x = 0, z)$$

$$+ \frac{i}{2} \partial_{xx} n(x = 0, z) \frac{|z_0 - z|}{k_0}. \tag{B5}$$

It may have appeared to be artificial to insert the amplitude factor $F(\mathbf{r}, \mathbf{r}_0)$ in the integral (3.9)—the derivation of Section 3 could have been carried out just as well by using $F(z, z_0)$ in (3.9). However, in that case the second term in (B4) would have been absent, and the final result (B5) would have contained an additional spurious contribution $-in(x = 0, z)/4k_0|z_0 - z|$.

Using (B5) in (3.14) and (3.15) gives for smooth perturbations

$$\delta\varphi = \frac{k_0}{2} \int_0^{z_0} n(x = 0, z) \, dz, \tag{B6}$$

$$\delta \ln A = -\frac{1}{4} \int_0^{z_0} |z_0 - z| \partial_{xx} n(x = 0, z) \, dz. \tag{B7}$$

Eq. (B6) gives the first-order phase shift of the wave: it is equal to the phase shift for the 1-D case given by (2.11). This shows that also in a 2-D medium the phase shift is handled correctly to first order by the Rytov approximation, even when this phase shift is not small compared to a period. Eq. (B7) gives a focusing integral that accounts for the first-order amplitude changes due to ray-geometrical focusing. It is similar to the focusing integrals derived previously (Snieder 1988; Neele, VanDecar & Snieder 1993).

APPENDIX C: THE TRAVELTIME PERTURBATION FOR NARROW-BAND TRANSIENTS

The change τ in the arrival time of narrow-band transient arrivals can be determined by requiring that the relation (4.3) is satisfied in the least-squares sense, i.e. by minimizing

$$M(\tau, \xi) \equiv \int |u_R(\mathbf{r}_0, t) - \xi u_0(\mathbf{r}_0, t - \tau)|^2 \, dt \tag{C1}$$

as a function of τ and ξ. Because of Parseval's theorem this is equivalent to minimizing

$$M(\tau, \xi) \equiv 2\pi \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} |u_R(\mathbf{r}_0, \omega) - \xi u_0(\mathbf{r}_0, \omega) e^{i\omega\tau}|^2 \, d\omega. \tag{C2}$$

Using (E2), (E3) and (E6) it follows with (E1) that

$$\frac{\partial F}{\partial s} = \frac{1}{v}\frac{\partial v}{\partial s}F. \tag{E7}$$

Note that the only difference from the corresponding equation (D7) for the 3-D case is a factor of 2 on the right-hand side of (D7). This difference arises because of the difference of a factor of 2 in the first term of the right-hand side for the expressions (D6) and (E6) for the 3-D and the 2-D cases respectively.

The solution of the differential equation (E7) can be found using the technique in the previous section. For a small distance η from s_0, this leads to eq. (8.7) rather than (7.2) for the 3-D case.