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The formation of caustics in two- and three-dimensional media

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SUMMARY

In terms of ray theory, the focus point (also related to caustics and triplications) is the point in space where the ray position is stationary for perturbations in the initial condition. Criteria for the formation of caustics are presented. With ray perturbation theory, a condition for the development of triplications is defined for plane wave sources and for point sources. This theory is then applied to two cases of slowness media: 1-D slowness perturbation models and 2-D Gaussian random media. The focus position in 1-D slowness models is proportional to the inverse of the square root of relative slowness fluctuations. For Gaussian random media, the distance at which caustics generate is dependent on the relative slowness perturbation to the power of $-2/3$. We use snapshots of propagating plane wavefields to show that caustics develop as predicted by theory. The theory for caustic formation can be generalized to three dimensions.

Key words: caustics, focus point, Gaussian random media, ray perturbation theory.

1 INTRODUCTION

In terms of ray theory, the concept of caustics is understood as the focus point in space through which rays go. The consequence of the generation of caustics in a wavefield is, in the ray geometrical limit, that the amplitude in the wavefield is infinitely high at the focus point because the geometrical spreading factor is zero at the caustic point (Aki & Richards 1980; Menke & Abbot 1990). This phenomenon has been investigated by several authors: White *et al.* (1988) used limit theorems for stochastic differential equations on the equation of dynamic ray tracing to predict when caustics start to develop in Gaussian random media, Kravtsov (1988) gave a thorough description of caustics, and Brown & Tappert (1986) used Chapman's method to write explicitly the variation of 2-D and 3-D wavefields in the vicinity of focus points. They set up three properties of transient wavefields away from caustics; the most important characteristic of transient waves through caustics is that the triplication will generate after the ballistic wavefield due to causality.

A new theory for caustic formation is presented. This theory is based on ray perturbation theory but is formally equivalent to dynamical ray theory as used in White *et al.* (1988) because the normal derivative of the equations in ray perturbation theory is identical to the equation of dynamic ray tracing (Pulliam & Snieder 1998). In contrast to the treatment of White *et al.* (1988), this application is not restricted to random media.

In Section 2, the general theory for the caustic formation of wavefields emitted by plane wave sources and point sources is presented. The theory is then applied on a 1-D slowness perturbation medium and a 2-D Gaussian random medium for both plane wave sources and point sources. The results for the 2-D Gaussian random medium are similar to those given in White *et al.* (1988). In Section 3, the theory for caustic formation is tested on numerical experiments where a plane wavefield propagates in a 1-D slowness perturbation field and in a 2-D Gaussian random medium.

2 THEORY

We demonstrate how the focal length of converging wavefields in 2-D slowness perturbation fields can be computed. First, we derive the general theory for two distinct source geometries, the plane wave (plw) source and the point source (ps). Second, we apply this theory to two case studies, 1-D slowness perturbation fields and 2-D Gaussian random media. The theory presented for caustic formation can be generalized to three dimensions.

2.1 General theory

We make use of ray perturbation theory (Snieder & Sambridge 1992) and separate the ray into a reference ray and a perturbed ray. The slowness field, $u=u_0+u_1$, is decomposed into the reference slowness field, u_0, and the slowness perturbation field, u_1. The reference slowness u_0 is kept constant in this study, which means that the reference ray is a straight line. The perpendicular deflection from the reference ray to the perturbed ray at propagation distance x_0 is denoted by $q(x_0)$.

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First, the case of an incoming plane wave is treated. Imagine two horizontal reference rays with slightly different initial positions. One reference ray is at the position z while the other reference ray is at the position $z+\delta z$; see Fig. 1 for a definition of the geometrical variables. For each reference ray there is a perturbed ray due to the slowness perturbation in the medium. The condition for caustics, that is, where the two perturbed rays intersect, gives the following equation:

$$q(x_0, z+\delta z) + \delta z - q(x_0, z) = 0 \tag{1}$$

or

$$\frac{\partial q}{\partial z}(x_0) = \lim_{\delta z \to 0} \frac{q(x_0, z+\delta z) - q(x_0, z)}{\delta z} = -1\,. \tag{2}$$

Snieder & Sambridge (1992) showed how the perpendicular ray deflection $q(x_0)$ from the reference ray can be computed given the slowness perturbation u_1:

$$q(x_0) = \int_0^{x_0} G(x_0, x)\left[\partial_\perp\left(\frac{u_1}{u_0}\right)\right](x)dx\,, \tag{3}$$

with $\partial_\perp$ the component of the gradient perpendicular to the reference ray, so that

$$q(x_0) = \int_0^{x_0} G(x_0, x)\frac{\partial}{\partial z}\left(\frac{u_1}{u_0}\right)(x)dx \tag{4}$$

for a horizontal reference ray. The Green's function

$$G(x_0, x) = \begin{cases} 0 & \text{if } x_0 < x \\ x_0 - x & \text{if } x_0 > x \end{cases} \tag{5}$$

has the boundary conditions $G(0, x) = \dot{G}(0, x) = 0$. The condition for caustics in eq. (2) contains the partial derivative of $q(x_0)$ with respect to z. Using eq. (4) together with the condition for caustics in eq. (2) at given z, we find that caustics are formed at x_0 when

$$\int_0^{x_0} G(x_0, x)\frac{\partial^2}{\partial z^2}\left(\frac{u_1}{u_0}\right)(x)dx = -1\,. \tag{6}$$

Second, the point source case is considered. We investigate the generation of caustics developing for rays that leave a point source with an azimuth φ. Assume again that two reference rays with slightly different initial positions are emitted from the source. One reference ray is sent in the direction $\varphi+\delta\varphi/2$, while the other reference ray is emitted in the direction $\varphi-\delta\varphi/2$. The distance between the reference rays is given by $x_0\delta\varphi$. The condition that the two perturbed rays cross each other leads to the following equation:

$$q\left(x_0, \varphi+\frac{1}{2}\delta\varphi\right) + x_0\delta\varphi - q\left(x_0, \varphi-\frac{1}{2}\delta\varphi\right) = 0 \tag{7}$$

or

$$\frac{1}{x_0}\frac{\partial q}{\partial \varphi}(x_0) = \frac{1}{x_0}\lim_{\delta\varphi \to 0} \frac{q\left(x_0, \varphi+\frac{1}{2}\delta\varphi\right) - q\left(x_0, \varphi-\frac{1}{2}\delta\varphi\right)}{\delta\varphi} = -1\,. \tag{8}$$

Using $\partial_\perp = (1/x)(\partial/\partial\varphi)$ in eq. (3), the perpendicular ray deflection to the reference ray is derived. Hence,

$$q(x_0) = \int_0^{x_0} G(x_0, x)\frac{1}{x}\frac{\partial}{\partial\varphi}\left(\frac{u_1}{u_0}\right)(x)dx\,. \tag{9}$$

The Green's function in eq. (9) for the reference ray with azimuth φ is the same as in the case of incoming plane waves, which is stated in eq. (5). With eq. (9) combined with the condition of caustics in eq. (8) at given z, we find that caustics generate at x_0 when

$$\frac{1}{x_0}\int_0^{x_0} G(x_0, x)\frac{1}{x}\frac{\partial^2}{\partial\varphi^2}\left(\frac{u_1}{u_0}\right)(x)dx = -1\,. \tag{10}$$

The second derivative of u_1/u_0 with respect to the transverse coordinate is an important quantity. It reflects the fact that it is the curvature of the relative slowness perturbation that generates caustics. For example, negative $\partial^2/\partial z^2(u_1/u_0)$ and $\partial^2/\partial\varphi^2(u_1/u_0)$ lead to the focusing of wavefields, whereas in areas with defocusing effects the two quantities are positive.

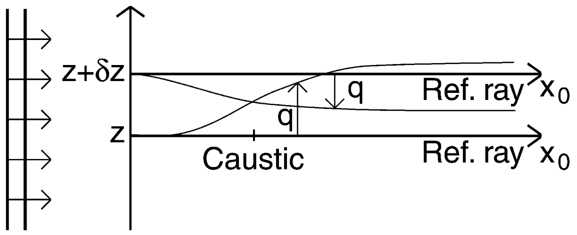

Figure 1. Definition of the geometric variables for an incoming plane wave in a 2-D medium with a constant reference slowness. There is one horizontal reference ray at z and another one at $z+\delta z$. The caustic develops at the intersection point of the two perturbed rays.

2.2 A medium with 1-D slowness perturbations

The focus position of a plane wave propagating in a medium with a constant reference slowness field u_0 and 1-D slowness perturbations $u_1(z)$ can be computed analytically. The reference ray in such a medium is a straight at given z. The condition for caustics in the case of incident plane waves given by eq. (6) can be used to determine when caustics start to generate at the offset $x_{\text{caus}}^{\text{plw}}$ at given z. The integration in eq. (6) is carried out from 0 to $x_{\text{caus}}^{\text{plw}}$. Hence,

$$x_{\text{caus}}^{\text{plw}}(z) = \sqrt{\frac{-2}{\frac{\partial^2}{\partial z^2}\left(\frac{u_1}{u_0}\right)(z)}}\,. \tag{11}$$

The focal distance $x_{\text{caus}}^{\text{ps}}$ of wavefields emitted by point sources is easily derived from the condition for caustics in eq. (10). The second derivative $\partial^2/\partial\varphi^2 = x^2\partial^2/\partial z^2$, which permits an evaluation of the integration in eq. (10) in the range 0–$x_{\text{caus}}^{\text{ps}}$. Thus,

$$x_{\text{caus}}^{\text{ps}}(z) = \sqrt{\frac{-6}{\frac{\partial^2}{\partial z^2}\left(\frac{u_1}{u_0}\right)(z)}}\,. \tag{12}$$

The distance between the source and receiver is denoted L. If $x_{\text{caus}}^{\text{plw/ps}}(z) < L$, triplications will be present in the recorded wavefield.

2.3 Gaussian random media

Next, we discuss the formation of caustics in Gaussian random media. The autocorrelation function $F(r)$ of a Gaussian random medium is given by

$$F(r) = \langle u_1(\mathbf{r}_1)u_1(\mathbf{r}_2)\rangle = (\varepsilon u_0)^2 \exp\left(-\frac{r^2}{a^2}\right), \tag{13}$$

where ε is the rms value of the relative slowness perturbations, a denotes the correlation length (or roughly the length scale of slowness perturbations) and $r=|\mathbf{r}_1-\mathbf{r}_2|$. Notice that the reference slowness is biased in a realization of a finite Gaussian random model (e.g. Müller *et al.* 1992). However, this artefact does not affect the derivatives of the slowness.

According to eq. (2), caustics develop in a plane wavefield when $\partial q/\partial z=-1$. This implies that on average in a random medium caustics develop when

$$\left\langle \left(\frac{\partial q}{\partial z}\right)^2 (x_0) \right\rangle = 1\,, \tag{14}$$

where $\langle \dots \rangle$ is the expectation value. For this reason the following quantity is used to monitor the formation of caustics:

$$H^{\mathrm{plw}}(x_0) \equiv \left\langle \left(\frac{\partial q}{\partial z}\right)^2 (x_0) \right\rangle = \frac{1}{u_0^2}\int_0^{x_0}\int_0^{x_0} G(x_0,x')G(x_0,x'') \times \left\langle \frac{\partial^2}{\partial z^2}u_1(x')\,\frac{\partial^2}{\partial z^2}u_1(x'')\right\rangle dx'dx''\,. \tag{15}$$

The monitor $H^{\mathrm{plw}}(x_0)$ is zero at the source position and $H^{\mathrm{plw}}(x_0)=1$ when caustics start to develop at the offset x_0 according to eq. (14)

We follow the same method as used in Roth *et al.* (1993) to evaluate the right-hand side of eq. (15). First, the expectation value of the slowness perturbation field differentiated with respect to z twice, at offsets x' and x'', in eq. (15) is expressed in a simple form containing the characteristic parameters for the Gaussian random medium. The following expression is evaluated on the horizontal reference ray z_0:

$$\left.\left\langle \frac{\partial^2}{\partial z^2}u_1(x',z)\,\frac{\partial^2}{\partial z^2}u_1(x'',z)\right\rangle\right|_{z=z_0} = \left.\left\langle \frac{\partial^4}{\partial z'^2\partial z''^2}u_1(x',z')u_1(x'',z'')\right\rangle\right|_{z'=z''=z_0} = \left.\frac{\partial^4 F(r)}{\partial z'^2\partial z''^2}\right|_{z'=z''=z_0}. \tag{16}$$

The autocorrelation function $F(r)$ is differentiated twice with respect to z' and z'' in eq. (16), which gives

$$\left.\frac{\partial^4 F(r)}{\partial z'^2\partial z''^2}\right|_{z'=z''=z_0} = \left.\frac{3}{r^2}\left(F''(r)-\frac{F'(r)}{r}\right)\right|_{z'=z''=z_0}. \tag{17}$$

The prime and double-prime of $F(r)$ signify a single and double differentiation with respect to r. Using the autocorrelation function $F(r)$ in eq. (13) for Gaussian random media, the left-hand side of eq. (16) is finally written as

$$\left.\left\langle \frac{\partial^2}{\partial z^2}u_1(x',z)\,\frac{\partial^2}{\partial z^2}u_1(x'',z)\right\rangle\right|_{z=z_0} = 12\frac{(\varepsilon u_0)^2}{a^4}\exp\left(-\left(\frac{r}{a}\right)^2\right). \tag{18}$$

The right-hand side of the monitor for plane waves in eq. (15) can be simplified further. Define

$$f(r) \equiv \left.\left\langle \frac{\partial^2}{\partial z^2}u_1(x',z)\,\frac{\partial^2}{\partial z^2}u_1(x'',z)\right\rangle\right|_{z=z_0}, \tag{19}$$

where $r=|x'-x''|$ and

$$\eta(x',x'') = G(x_0,x')G(x_0,x'') = x_0^2 + x'x'' - x_0(x'+x'')\,. \tag{20}$$

We then derive from eq. (15)

$$\int_0^{x_0}\int_0^{x_0} G(x_0,x')G(x_0,x'')\left.\left\langle \frac{\partial^2}{\partial z^2}u_1(x')\,\frac{\partial^2}{\partial z^2}u_1(x'')\right\rangle\right|_{z=z_0} dx'dx'' = \int_0^{x_0}\int_0^{x_0}\eta(x',x'')f(|x'-x''|)dx'dx'' \tag{21}$$

for x' and x'' smaller than x_0. Using the integration technique in Roth *et al.* (1993), the expression for the monitor in eq. (15) is simplified further. The details of this integration method are explained in Appendix A; here we just give the results. The double integration in eq. (21) from 0 to x_0 is changed to an integration from 0 to x_0 of the function $f(r)$ in eq. (19) multiplied by a summation of two integrations of $\eta(x', x'')$ in eq. (20) from r to x_0 and from 0 to x_0-r, respectively. In brief, the right-hand side of eq. (21) is written as

$$\int_0^{x_0} dr f(r)\left[\int_r^{x_0}\eta(x',x'-r)dx' + \int_0^{x_0-r}\eta(x',x'+r)dx'\right]. \tag{22}$$

The solution to the two integrations of $\eta(x', x'')$ inside the square brackets are computed analytically:

$$\int_r^{x_0}\eta(x',x'-r)dx' = \int_0^{x_0-r}\eta(x',x'+r)dx' = \frac{1}{3}x_0^3 - \frac{1}{2}x_0^2 r + \frac{1}{6}r^3\,. \tag{23}$$

The expressions for the function $f(r)$ in eq. (19) and for the integration of $\eta(x', x'')$ in eq. (23) are used together with the expression for the monitor in eq. (15). Hence, the monitor for plane waves propagating in a Gaussian random medium simplifies to

$$H^{\mathrm{plw}}(x_0) = 12\frac{\varepsilon^2}{a^4}\int_0^{x_0}\left(\frac{2}{3}x_0^3 - x_0^2 r + \frac{1}{3}r^3\right)\exp\left(-\left(\frac{r}{a}\right)^2\right)dr\,. \tag{24}$$

By letting x_0 go to zero in eq. (24) it is easy to verify that $H^{\mathrm{plw}}(0)=0$.

Assume first that the propagation length is less than the correlation length, i.e. $x_0/a<1$. The exponential function is set to unity in this regime and the integration of the right-hand side of eq. (24) is carried out directly. Hence,

$$H^{\mathrm{plw}}(x_0) = 3\varepsilon^2 \left(\frac{x_0}{a}\right)^4 \ll 1\,, \tag{25}$$

which reflects the fact that caustics are not formed in this regime.

Suppose instead that the propagation distance is much greater than the correlation length, i.e. $x_0/a \gg 1$. We can then compute the analytical solution of the monitor in eq. (24) by letting the range of integration go to infinity because the exponential in the integrand approaches zero for $r \gg a$. Thus,

$$\begin{aligned} H^{\mathrm{plw}}(x_0) &\approx 12\,\frac{\varepsilon^2}{a^4}\int_0^\infty \left(\frac{2}{3}\,x_0^3 - x_0^2 r + \frac{1}{3}\,r^3\right)\exp\left(-\left(\frac{r}{a}\right)^2\right)dr \\ &= 12\,\frac{\varepsilon^2}{a^4}\left(\frac{\sqrt{\pi}ax_0^3}{3} - \frac{a^2x_0^2}{2} + \frac{a^4}{6}\right) \\ &\approx 4\sqrt{\pi}\varepsilon^2\left(\frac{x_0}{a}\right)^3. \end{aligned} \tag{26}$$

We have made use of the assumption that $x_0/a \gg 1$ to eliminate the last two terms in brackets of eq. (26). Let L denote the source–receiver offset. We then derive the non-dimensional number L/a from eq. (26) in the case where caustics develop at $x_0<L$. Hence, by using $H^{\mathrm{plw}}(L) \geq 1$, we obtain

$$\frac{L}{a} \geq \frac{\varepsilon^{-2/3}}{(4\sqrt{\pi})^{1/3}} = 0.52\varepsilon^{-2/3}\,. \tag{27}$$

For a point source the generation of caustics can be evaluated along similar lines. The monitor $H^{\mathrm{ps}}(x_0)$ is defined in the same way as the monitor for plane waves except that the condition for the formation of caustics in eq. (8) for point sources is applied. Thus,

$$\begin{aligned} H^{\mathrm{ps}}(x_0) &\equiv \left\langle \left(\frac{1}{x_0}\frac{\partial q}{\partial \varphi}\right)^2 (x_0) \right\rangle \\ &= \frac{1}{x_0^2 u_0^2}\int_0^{x_0}\int_0^{x_0} G(x_0,x')G(x_0,x'') \\ &\quad \times \frac{1}{x'x''}\left\langle \frac{\partial^2}{\partial\varphi^2}\,u_1(x')\,\frac{\partial^2}{\partial\varphi^2}\,u_1(x'')\right\rangle dx'dx''\,. \end{aligned} \tag{28}$$

According to eq. (8), caustics develop at the offset x_0 when the monitor in eq. (28) is equal to 1. The mean value

$$\left\langle \frac{\partial^2}{\partial\varphi^2}\,u_1(x')\,\frac{\partial^2}{\partial\varphi^2}\,u_1(x'')\right\rangle$$

is related to

$$\left\langle \frac{\partial^2}{\partial z^2}\,u_1(x')\frac{\partial^2}{\partial z^2}\,u_1(x'')\right\rangle$$

by using the chain rule:

$$\frac{\partial}{\partial z} = \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi} = \frac{1}{x}\frac{\partial}{\partial \varphi}\,, \tag{29}$$

because $z=x\varphi$ for small values of φ. Thus,

$$\begin{aligned} &\left\langle \frac{\partial^2}{\partial\varphi^2}\,u_1(x')\,\frac{\partial^2}{\partial\varphi^2}\,u_1(x'')\right\rangle \\ &\quad = (x'x'')^2\left\langle \frac{\partial^2}{\partial z^2}\,u_1(x',z)\,\frac{\partial^2}{\partial z^2}\,u_1(x'',z)\right\rangle\Bigg|_{z=z_0}. \end{aligned} \tag{30}$$

The procedure used for the derivation of the monitor for plane waves is repeated for the monitor for point sources. The only difference from the previous example is that the function $\eta(x',x'')=G(x_0,x')G(x_0,x'')x'x''=x_0^2x'x''+(x'x'')^2-x_0(x'+x'')x'x''$. The final result of the rather long derivation of the monitor for caustic formation in the point source case is given by

$$\begin{aligned} H^{\mathrm{ps}}(x_0) =& 12\,\frac{\varepsilon^2}{a^4}\int_0^{x_0}\left(\frac{1}{15}\,x_0^3 - \frac{4}{3}\,x_0r^2 + \frac{4}{3}\,r^3 + \frac{7}{15}\frac{r^5}{x_0^2}\right) \\ &\times \exp\left(-\left(\frac{r}{a}\right)^2\right)dr\,. \end{aligned} \tag{31}$$

By letting x_0 go to zero it can be shown that $H^{\mathrm{ps}}(0)=0$.

Similarly to eq. (25), it can easily be shown that triplications due to point source wavefields do not generate when the length scale of heterogeneity is greater than the source–receiver distance. Assume instead that $x_0/a \gg 1$ and carry on exactly as in the case of incident plane waves. The analytical expression of the right-hand side of eq. (31) is given by

$$H^{\mathrm{ps}}(x_0) = \frac{2\sqrt{\pi}}{5}\,\varepsilon^2\left(\frac{x_0}{a}\right)^3. \tag{32}$$

The non-dimensional number L/a for the condition that caustics develop in the recorded wavefield is derived from eq. (32); the condition $H^{\mathrm{ps}}(L)\geq 1$ thus gives

$$\frac{L}{a} \geq \left(\frac{5}{2\sqrt{\pi}}\right)^{1/3}\varepsilon^{-2/3} = 1.12\varepsilon^{-2/3}\,. \tag{33}$$

It is instructive to compare eq. (27) for plane waves and eq. (33) for point sources with the estimates obtained by White *et al.* (1988). They used limit theorems for stochastic differential equations on the equation of dynamic ray tracing in Gaussian random media to calculate the probability that a caustic occurs at a certain propagation distance. In Figs 4 and 5 of White *et al.* (1988), they demonstrate universal curves for the probability of caustic formation as a function of the universal distance, defined as

$$\tilde{\tau} = (8\pi)^{1/6}\varepsilon^{2/3}\,\frac{L}{a}\,, \tag{34}$$

where we have made a change of symbol from White *et al.* (1988) by using ε for the relative rms value of slowness perturbations and L for the propagation distance of the wavefield. This means that in the theory of White *et al.* (1988), caustics develop when the non-dimensional number L/a is given by

$$\frac{L}{a} = \frac{\tilde{\tau}}{(8\pi)^{1/6}}\,\varepsilon^{-2/3}\,. \tag{35}$$

This expression has the same dependence on ε as the condition for caustics in eqs (27) and (33).

According to Figs 4 and 5 of White *et al.* (1988), the highest probability for the generation of caustics for plane waves is found for $\tilde{\tau}=0.9$ and for point sources $\tilde{\tau}=1.9$. By inserting the appropriate value of $\tilde{\tau}$ into the factor $\tilde{\tau}/(8\pi)^{1/6}$ from eq. (35), we find that the factors are 0.53 and 1.11 for the cases of plane waves and point sources, respectively. Comparing these two numbers with the corresponding factors in eqs (27) and (33), we see that there is a good agreement between the work of White *et al.* (1988) and our work.

Although we have derived the condition for caustics due to plane waves and point sources in two dimensions, the theory for caustics can be generalized to three dimensions. In three dimensions, the equation for the ray perturbations q_1 and q_2 in the directions perpendicular to the ray decouples for a homogeneous reference model and a coordinate system that does not rotate around the reference ray (see eq. 50 of Snieder & Sambridge 1992). The condition for caustics in eqs (2) and (8) for plane waves and point sources, respectively, can be applied to the ray perturbation in two orthogonal directions separately. For example, as shown in Appendix B, the non-dimensional number L/a for the point focus in 3-D Gaussian random media is given by eq. (27) for plane waves and by eq. (33) for point sources. Notice that in three dimensions a caustic is not necessarily the same as a point focus. A caustic can in that case also be a line of focus points, whereas a focus point, as the words imply, is located at a point.

3 NUMERICAL EXAMPLES

In this section, numerical examples of caustic formation of plane wavefields are shown for two distinct media: a 1-D medium with the slowness perturbation field described by $u_1(z) = \sqrt{2}\varepsilon u_0 \sin[(z+z_0)^4/k]$, and a 2-D Gaussian random medium with the slowness perturbation field described by eq. (13). The quantity u_0 is the reference slowness, which is constant for all numerical experiments shown in this paper. The rms value of relative slowness fluctuations is denoted by ε. The parameters z_0, k and ε are adjusted such that the development of triplications in the media is significant.

In Fig. 2(a), the 1-D slowness medium with $z_0 = 350$ m, $k = 1.5 \times 10^{10}$ m^4, $u_0 = 2.5 \times 10^{-4}$ s m^{-1} and $\varepsilon = 0.035$ is plotted. It is seen that the slowness field changes more and more rapidly with increasing z. In Fig. 2(b), the focal distance of a plane wavefield propagating in the 1-D slowness medium shown in (a) is computed using eq. (11). The offset from the source position is plotted on the abscissa while the depth at which caustics start to develop is plotted on the ordinate. The focal distance of the converging plane wavefield is shown as a solid line. Notice that there are zones with defocusing of the plane wavefield between 120 and 205 m, between 265 and 305 m, between 350 and 380 m, etc. In these zones the wavefield propagates through a zone with a positive curvature of the relative slowness perturbation, so caustics do not develop. The focal distance is thus infinite. The curvature of the relative slowness fluctuations increases with increasing z, so the focal distance of the converging wavefield decreases as depth increases.

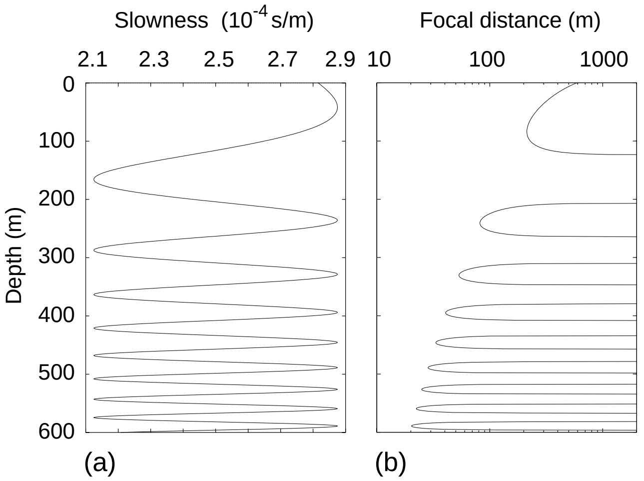

Figure 2. (a) The 1-D slowness field with $\varepsilon = 0.035$. (b) The focal distance (solid line) of a plane wavefield calculated as a function of depth. Notice that the incoming plane wavefield is being focused in regions with positive slowness perturbations and defocused when the slowness perturbation is negative.

In Fig. 3, snapshots of a plane wavefield propagating through the 1-D slowness perturbation field with the same values of z_0, k, u_0 and ε as for the 1-D medium in Fig. 2(a) are shown. The snapshots are produced with a finite difference solution of the acoustic wave equation. The central frequency is 1000 Hz and the applied source function is a Ricker wavelet. The snapshots are taken every 5 ms, with the first snapshot at the source position and the last snapshot at about 100 m offset. Positive amplitudes are dark while negative amplitudes are light. The first triplications are visible in the snapshot at $t = 10$ ms (~ 40 m) at depths below 500 m as the wavefield contains large positive amplitudes. In the snapshots for $t = 15$ ms (~ 60 m), 20 ms (~ 80 m) and 25 ms (~ 100 m), the triplications generated in the wavefield are very clear as they give rise to a half-bowtie-form wave after the ballistic wavefront. Comparing the theoretically predicted minimum focal distance of the converging wavefield in Fig. 2(b) with the offset at which triplications start to develop in the wavefield in Fig. 3, we find that there is good agreement between the theory presented for caustic formation and the numerical 1-D experiment.

In Fig. 4, snapshots of a plane wave propagating in a 2-D Gaussian random medium with reference slowness $u_0 = 2.5 \times 10^{-4}$ s m^{-1}, relative slowness fluctuation $\varepsilon = 0.025$ and correlation length $a = 7.1$ m are presented. The central frequency is 1000 Hz, while the Ricker wavelet is applied as a source impulse. The 10 snapshots are computed for every 2.5 ms, where the first snapshot is taken at the initial wavefront and the last snapshot is taken at about 90 m offset. In the upper and lower parts of the plane wavefronts in Fig. 4, a circular wave due to diffraction at the edge of the numerical grid can be seen. Inserting the appropriate value for ε into eq. (27), the non-dimensional number $L/a = 6.1$ for the development of triplications in the Gaussian random medium is found. The expectation value of the offset at which caustics start to generate is then $L = 43$ m for $a = 7.1$ m. In Fig. 4, no triplications are observed in the wavefront at $t = 0$, 2.5 or 5 ms, i.e. at approximately 0, 10 and 20 m, respectively. Then, for $t = 7.5$ ms (~ 30 m) and 10 ms (~ 40 m), the multipathing that is associated with the formation of caustics can be seen as a minor wavefield after the ballistic wavefront. This generation of minor wavefields, delayed compared to the ballistic wavefront, is due neither to uncertainties in the finite difference code or to scattering effects (for the employed wave $\lambda/a \approx 0.5$) but instead to caustic formation. For the last five snapshots at $t = 12.5$ ms (~ 50 m), 15 ms (~ 60 m), 17.5 ms (~ 70 m), 20 ms (~ 80 m) and 22.5 ms (~ 90 m), triplications are developing rather strongly after the wavefront. The maximum amplitude variation along the wavefield for each wavefront is plotted with as a solid white line

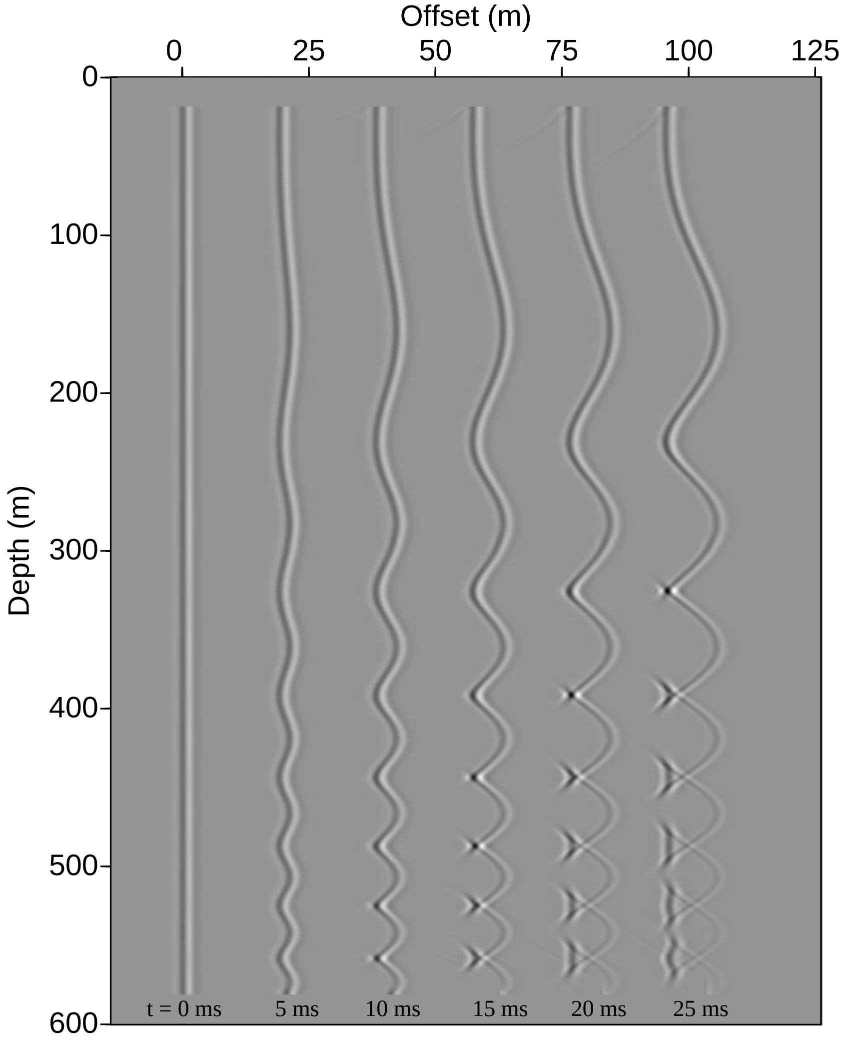

Figure 3. Snapshots of plane wave propagation in the 1-D slowness perturbation model with $\varepsilon = 0.035$. The absolute traveltimes $t = 0$, 5, 15, 20 and 25 ms are marked at the respective wavefronts. Caustics become very clear after the ballistic wavefronts for $t = 15$, 20 and 25 ms.

in Fig. 4. For the initial wavefront at $t = 0$ ms, the amplitude is constant, while the maximum amplitude along the wavefield varies with increasing extrema for the wavefronts for larger t. The bar in the upper-right corner of Fig. 4 shows the relative percentage of the amplitude variations in the perturbed slowness model compared with the reference amplitude computed for the homogeneous reference slowness model. Notice that the largest positive values of the maximum amplitude along the wavefronts correspond to the parts of the wavefronts with darkest shades, while the negative amplitude variations are shown with light shades. Witte *et al.* (1996) uses the kinematic ray tracing equation to construct a ray diagram for a Gaussian random medium with fixed $\varepsilon = 0.03$, but with different values of the correlation length a. Using eq. (27) with $\varepsilon = 0.03$ gives the non-dimensional number $L/a = 5.4$. Looking at the top panel in Fig. 4 of Witte *et al.* (1996) for $L/a = 10$, it is seen that the first caustics generate at $z \approx 5$–6, which corresponds well with the theoretical value computed with eq. (27).

4 CONCLUSIONS

In this paper we have developed a theory for the formation of caustics. The theory is based on ray perturbation theory, but is equivalent to a similar approach by White *et al.* (1988) where the equation of dynamic ray tracing is used to predict when triplications develop in Gaussian random media.

We have applied the theory for the generation of caustics in two case studies (that is, 1-D slowness perturbations fields and 2-D Gaussian random media) where the plane wave source and the point source are taken into account. The theory for caustic formation can be generalized to wavefields propagating in three dimensions. We find that the formation of caustics for 1-D slowness perturbation fields depends on the inverse of the square root of the relative slowness perturbation, while for Gaussian random media the formation of caustics is dependent upon the relative slowness perturbation to the power of $-$two-thirds.

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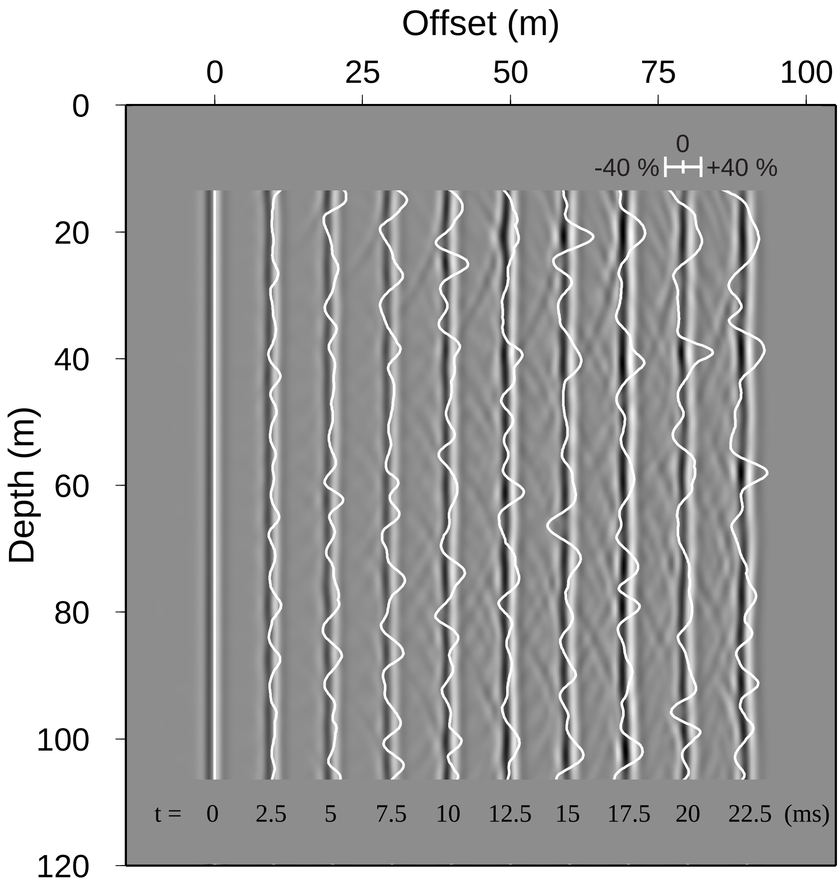

Figure 4. Snapshots of plane wave propagation in the 2-D Gaussian random model with $\varepsilon = 0.025$ and $a = 7.1$ m. The absolute traveltimes $t = 0$, 2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20 and 22.5 ms are marked at the respective wavefronts. Caustics develop in the wavefields for $t \geq 7.5$ ms. The maximum amplitude variation along the wavefield for each wavefront is shown as a solid white line. Notice that the shade in the wavefronts gets darker when the maximum amplitude is at a peak. The bar in the upper-right corner shows the percentage variation of the maximum amplitude in the perturbed slowness model compared with the reference amplitude computed in the constant reference slowness model.

REFERENCES

Aki, K. & Richards, P.G., 1980. *Quantitative Seismology: Theory and Methods,* W. H. Freeman, San Francisco, CA.

Brown, M. & Tappert, F.D., 1986. Causality, caustics, and the structure of transient wave fields, *J. acoust. Soc. Am.,* **80,** 251–255.

Kravtsov, Y.A., 1988. Rays and caustics as physical objects in *Prog. in Optics,* Vol. XXVI, pp. 227–348, ed. Wolf, E., Elsevier, Amsterdam.

Menke, W. & Abbot, D., 1990. *Geophysical Theory,* Columbia University Press, NY.

Müller, G., Roth, M. & Korn, M., 1992. Seismic-wave traveltimes in random media, *Geophys. J. Int.,* **110,** 29–41.

Pulliam, J. & Snieder, R., 1998. Ray perturbation theory, dynamic ray tracing and the determination of Fresnel zones, *Geophys. J. Int.,* **135,** 463–469.

Roth, M., Müller, G. & Snieder, R., 1993. Velocity shift in random media, *J. geophys. Res.,* **115,** 552–563.

Snieder, R. & Sambridge, M., 1992. Ray perturbation theory for traveltimes and ray paths in three-dimensional heterogeneous media, *Geophys. J. Int.,* **109,** 294–322.

White, B., Nair, B. & Baylies, A., 1988. Random rays and seismic amplitude anomalies, *Geophysics,* **53,** 903–907.

Witte, O., Roth, M. & Müller, G., 1996. Ray tracing in random media, *Geophys. J. Int.,* **124,** 159–169.

APENDIX A: DERIVATION OF EQ. (22)

In this appendix, the step from eq. (21) to eq. (22) is demonstrated. The right-hand side of eq. (21) is written as

$$\int_0^{x_0}\int_0^{x_0} \eta(x', x'') f(|x' - x''|) dx' dx'' , \tag{A1}$$

where

$$f(|x' - x''|) = \left\langle \frac{\partial^2}{\partial z^2} u_1(x', z) \, \frac{\partial^2}{\partial z^2} u_1(x'', z) \right\rangle \Bigg|_{z=z_0} \tag{A2}$$

and

$$\eta(x', x'') = G(x_0, x') G(x_0, x'') . \tag{A3}$$

The integration over x'' in eq. (A1) is split into one integration over x'' from 0 to x' and another integration over x'' from x' to x_0. Thus, eq. (A1) is rewritten as

$$\int_0^{x_0} dx' \left[\int_0^{x'} \eta(x', x'') f(x' - x'') dx'' + \int_{x'}^{x_0} \eta(x', x'') f(x'' - x') dx'' \right] . \tag{A4}$$

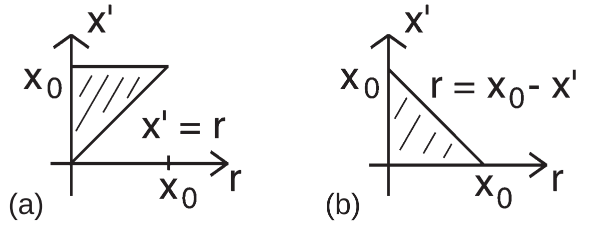

Figure A1. The integration technique given by Roth *et al.* (1993). Area of integration for (a) $\int_0^{x'} \eta(x', x'-r) f(r) dr$ and (b) $\int_0^{x_0-x'} \eta(x', x'+r) f(r) dr$ in eq. (A6).

Now define $r=x'-x''$ and $r=x''-x'$ for the first and second integrations over x'' in the brackets of eq. (A4) and carry out a change of variables in the two integrations over x'' inside the brackets. The result is given by

$$\int_0^{x_0} dx' \left[-\int_{x'}^{0} \eta(x', x'-r) f(r) dr + \int_0^{x_0-x'} \eta(x', x'+r) f(r) dr \right] \tag{A5}$$

or

$$\int_0^{x_0} dx' \left[\int_0^{x'} \eta(x', x'-r) f(r) dr + \int_0^{x_0-x'} \eta(x', x'+r) f(r) dr \right]. \tag{A6}$$

The integration over x' multiplied with the first and second integrations over r in the brackets of eq. (A6) corresponds to the triangle in the r–x' plane as shown in Figs A1(a) and (b), respectively. By changing the order of integration in eq. (A6), but still keeping in mind that the double-integration over r and x' must be over the triangles as shown in Figs A1(a) and (b), it is possible to rewrite eq. (A6) in the following way:

$$\int_0^{x_0} dr \left[\int_r^{x_0} \eta(x', x'-r) f(r) dx' + \int_0^{x_0-r} \eta(x', x'+r) f(r) dx' \right]. \tag{A7}$$

After rearranging the term $f(r)$ outside the integration over x', the result is finally given by

$$\int_0^{x_0} dr f(r) \left[\int_r^{x_0} \eta(x', x'-r) dx' + \int_0^{x_0-r} \eta(x', x'+r) dx' \right], \tag{A8}$$

which is the equation given in (22).

APPENDIX B: CAUSTIC FORMATION IN THREE DIMENSIONS

Imagine that a plane wavefield is propagating along the x-axis with the decoupled ray deflections q_1 and q_2 parallel to the y- and z-axes, respectively; see Fig. B1 for the experimental set-up. Using the results from Snieder & Sambridge (1992), the decoupled differential equations for the ray deflection coordinates are then given by

$$\frac{d^2}{dx^2} q_i = \hat{q}_i \cdot \nabla \left(\frac{u_1}{u_0} \right), \tag{B1}$$

where $i=1, 2$. The ray deflections are gathered together in the ray deflection vector $\mathbf{q}=(0, q_1, q_2)$. The condition for

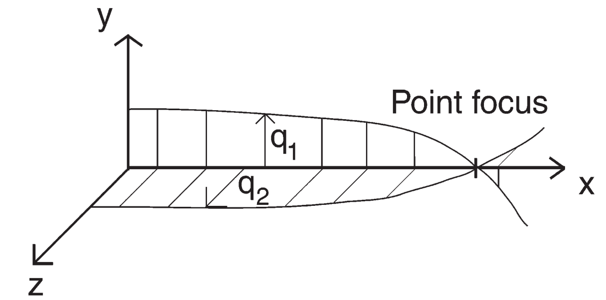

Figure B1. Estimation of the point focus in a 3-D medium.

caustic formation in eq. (2) is applied on each ray deflection coordinate. Hence,

$$\nabla \mathbf{q}(x_0) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \tag{B2}$$

for a point focus at offset x_0. In order to determine when caustics develop in a 3-D Gaussian random medium, the expectation value of $\nabla\mathbf{q}(x_0)\cdot\nabla\mathbf{q}(x_0)$ is computed. Thus, according to eq. (B2) for a point focus, caustics develop when

$$\begin{aligned} \langle \nabla\mathbf{q} \cdot \nabla\mathbf{q} \rangle (x_0) &= \left\langle \left(\frac{\partial q_1}{\partial y} \right)^2 (x_0) + \left(\frac{\partial q_2}{\partial z} \right)^2 (x_0) \right\rangle \\ &= \left\langle \left(\frac{\partial q_1}{\partial y} \right)^2 (x_0) \right\rangle + \left\langle \left(\frac{\partial q_2}{\partial z} \right)^2 (x_0) \right\rangle = 2 \end{aligned} \tag{B3}$$

at offset x_0. For 2-D Gaussian random media, the following result is derived:

$$H^{\mathrm{plw}}(x_0) \equiv \left\langle \left(\frac{\partial q}{\partial z} \right)^2 (x_0) \right\rangle = 1 \tag{B4}$$

when a caustic develops at offset x_0. According to eq. (26), $H^{\mathrm{plw}}(x_0)=4\sqrt{\pi}\varepsilon^2(x_0/a)^3$ for $x_0/a \gg 1$. This result can be used for each ray deflection q_i separately, so the monitors $H_1^{\mathrm{plw}}(x_0)$ and $H_2^{\mathrm{plw}}(x_0)$ for q_1 and q_2, respectively, are defined as

$$H_1^{\mathrm{plw}}(x_0) \equiv \left\langle \left(\frac{\partial q_1}{\partial y} \right)^2 (x_0) \right\rangle \quad \text{and}$$

$$H_2^{\mathrm{plw}}(x_0) \equiv \left\langle \left(\frac{\partial q_2}{\partial z} \right)^2 (x_0) \right\rangle, \tag{B5}$$

where $H_1^{\mathrm{plw}}(x_0)=H_2^{\mathrm{plw}}(x_0)=4\sqrt{\pi}\varepsilon^2(x_0/a)^3$. Combining eq. (B3) with the monitors defined in eq. (B5), we compute

$$H_1^{\mathrm{plw}}(x_0) + H_2^{\mathrm{plw}}(x_0) = 2 \tag{B6}$$

or

$$4\sqrt{\pi}\varepsilon^2 \left(\frac{x_0}{a} \right)^3 = 1 \tag{B7}$$

for a caustic at offset x_0 in a 3-D Gaussian random medium, which is also the result found for 2-D Gaussian random media. Similarly, the result for caustic formation due to a point source in 3-D Gaussian random media is unaltered from the result in the 2-D case.