

The Anatomy of Inverse Problems

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A major task of geophysics is to make quantitative statements about the interior of the earth. For this reason, inverse problems are an important area of geophysical research and industrial application. Figure 1 shows how many texts present inverse problems. The earth model is an element of a mathematical space that contains all allowable parameterizations of the earth's properties (or at least those properties relevant to a given experiment); this space is referred to as *model space*. The physics of the problem determines which data d correspond to a given model m . The problem of computing the model response (synthetic “data”) given a model is called the *forward problem*. The corresponding data reside in a mathematical space that is called *data space*. In many applications, one records the data, and the goal is to find the corresponding model. The task is called the *inverse problem*, as shown in Figure 1.

Unfortunately, Figure 1 is wrong. There is a simple reason for this. In general the model that one seeks is a continuous function of the space variables with infinitely many degrees of freedom. For example, the 3-D velocity structure in the earth has infinitely many degrees of freedom. On the other hand, the data space is always of finite dimension because any real experiment can only result in a finite number of measurements. A simple count of variables shows that the mapping from the data to a model cannot be unique; or equivalently, there must be elements of the model space that have no influence on the data. This lack of uniqueness is apparent even for problems involving idealized, noise-free measurements. The problem only becomes worse when the uncertainties of real measurements are taken into account. Although the uniqueness question is a hotly debated issue in the mathematical literature on inverse problems, it is largely irrelevant for practical inverse problems because they invariably have a nonunique solution (if by *solution* we mean an earth model). It is this nonuniqueness that makes Figure 1 deceptive, because the arrow pointing from data space to model space suggests that a unique model corresponds to every data set.

A more realistic scheme of inverse problems is shown in Figure 2. Given a model m , the physics of the problem determines the predicted data d ; this is called the *forward problem*. For a given data set, one determines an estimated model \hat{m} . We

refer to this as the *model estimation problem*. (Later, we will consider the generalization to the problem of estimating properties of models, rather than the models themselves.) Note that there may be many reasonable model estimates for a given data set and that the estimation procedure may be nonlinear even when the forward problem is linear. Thus, the mean of a set of numbers is a linear function of the numbers, whereas the median is a nonlinear function. Yet both the median and the mean may be reasonable estimators of the “center” of the set of numbers. Part of the art of solving inverse problems comes from the need to define what it means for an estimate to be reasonable.

Since the mapping from data space to model space is nonunique, the estimated model may also depend on the details of the algorithm that one has used for the estimation problem as well as on the regularization and model parameterization that has been used. In general, the estimated model \hat{m} differs from the true model m . For example, in seismic inversion the estimated model may be a blurred version of the true model. In addition, the data are always contaminated with errors; these errors represent an additional source of discrepancy between the estimated model and the true model.

One is not finished when the estimated model is constructed; it is essential to somehow quantify the error between the estimated model and the true model. This is called the *appraisal problem*. In this problem, one determines the uncertainty in the estimated model. This uncertainty has a statistical component related to the propagation of errors in the data, and a deterministic component that accounts for the finite resolution that is attained in the model estimation, as well as systematic errors in the problem.

For linear inverse problems, resolution kernels and confidence set analysis are powerful tools for formulating the appraisal problems, and the theory of linear error propagation is sufficiently well developed to account for the errors in the estimated model due to the errors in the data. For nonlinear inverse problems, the only tool available for the appraisal problem may be Bayesian inversion where one estimates the statistical properties of the model when the data and other knowledge are combined in a statistical sense by repeated sampling of the

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model space (e.g., Mosegaard and Tarantola, 1995; Gouveia and Scales, 1998). These techniques are only applicable to practical problems where the number of parameters is relatively small. This is partly due to computational cost of such methods, but is also related in a very subtle way to the behavior of probabilities in high dimensional spaces (see Scales and Tenorio, 2001). For large-scale inverse problems such as the determination of 3-D earth structure, Monte Carlo sampling methods are not feasible. This means that there is presently no operational theory to account for the appraisal problem of nonlinear inverse problems with large number of parameters (Snieder, 1998). Developing such a theory is a theoretical and practical challenge that is much more important than establishing uniqueness proofs of idealized mathematical problems.

In practice, one solves inverse problems with a certain goal. For example, one may use an estimated model obtained from an inversion of seismic data as a basis for deciding where to drill or how to optimally exploit a reservoir. In practice, one is never interested in the seismic velocity at a certain spot in the subsurface, but for a seismic interpreter it is crucial to know whether at a certain place a syncline or an anticline is present. This means that for practical inverse problems one is interested in patterns that can be used in a meaningful way for making decisions. In practice, these decisions are usually not based exclusively on the estimated model \hat{m} , but involve the integration of other data as well as human expertise. In addition, the uncertainty in the estimated model is an important factor in making decisions. Thus, Figure 3 gives a more realistic view on inverse problems, because it shows explicitly that decisions rather than model estimation is the endpoint of practical inverse problems.

It is interesting to consider the relation between the appraisal problem and the process of decision making. An important aspect of the appraisal problem is the statistical treatment of error propagation. This means that the appraisal problem has probability as an important component. Decisions are usually

based on risk rather than probability. This may be economic risk (where to drill), environmental risk (what happens when pollutants spread), or even academic risk (am I sure enough to publish my results). Risk is always concerned with probability plus another component such as profit or environmental impact. This means that in this stage those who produce the models and their uncertainty must interface with others for making decisions effectively.

Figure 3 offers an overview of the landscape of geophysical inversion. Those working on wave propagation problems focus on the forward problem. Those focused on the development of migration algorithms work on the estimation problem. Statisticians and a limited number of scientists in the inverse problem community are concerned with the appraisal problem. Seismic interpretation is usually based on an estimated model, and so can be thought of as a decision-making problem. It is illuminating to see how different researchers work on different parts of this problem. One may wonder whether these research efforts could be more effective when their activities are seen in the context of the anatomy of the inverse problem as shown in Figure 3.

In Figure 3, the estimated model \hat{m} forms an essential part of the inversion process. But is it necessary or desirable to produce an estimated model in the process of inversion of data? We, of course, are conditioned to produce models from our data. However, given the fact that this estimated model differs from the true model, one can be led astray by features in the estimated model that are artifacts of the inversion process. Another view on inverse problems is given in Figure 4 where the goal is to determine the range of models that are consistent with the data (as well as other information). This range of models can simply be a box within which all the models are believed to fit the data (there being no comparative relation among the models in the box), or it could be a probability distribution on the space of models $P(m)$ (in which case one can speak of the best or most probable model). Both the box and the probability are determined by the data, the uncertainties and whatever other data-independent information is available. As we have discussed previously (Scales and Snieder, 1997), people taking the former approach are called frequentists (this includes most statisticians), whereas people taking the latter approach are called Bayesians (this includes most geophysicists). In either case, the estimation and appraisal problem is replaced by the *inference problem*. Inference means characterizing somehow the set of models that explain the data (and satisfy whatever other information is available). One can then use this box, or the probability distribution $P(m)$, as a basis for making decisions. The tutorial by Scales and Tenorio (2001) shows a nice example of a “toy” inverse problems tackled from a Bayesian and a frequentist point of view.

The reader may be put off by the idea of solving inverse problems without constructing a model. Our minds are conditioned to making models from data. However, we have seen that these estimated models can only be the endpoint of research when one ignores the fact that models are being produced with the goal of making decisions. When seen in this larger context (Figures 3 or 4), it is worth considering whether one is not better served by knowing the probability distribution of the set of models than by knowing a single estimated model and a measure of its uncertainty.

It is interesting to consider how we would use a probability density function in a high-dimensional model space. The

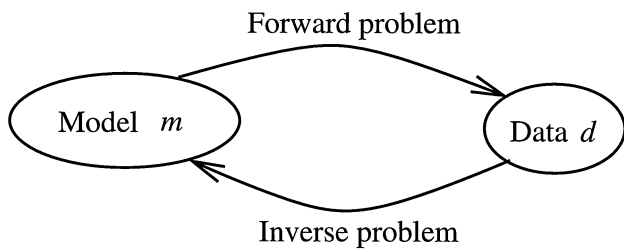


FIG. 1. The conventional view of inverse problems: find the model that predicts the measurements.

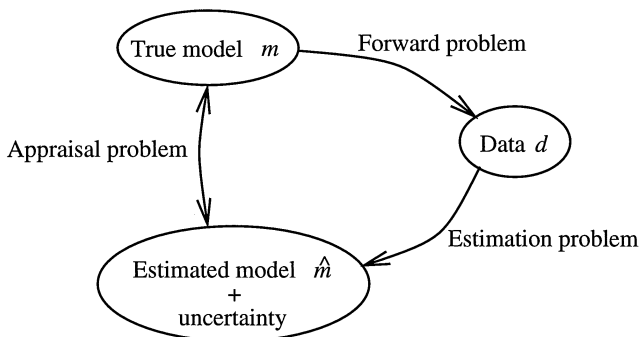


FIG. 2. An improved view of inverse problems.

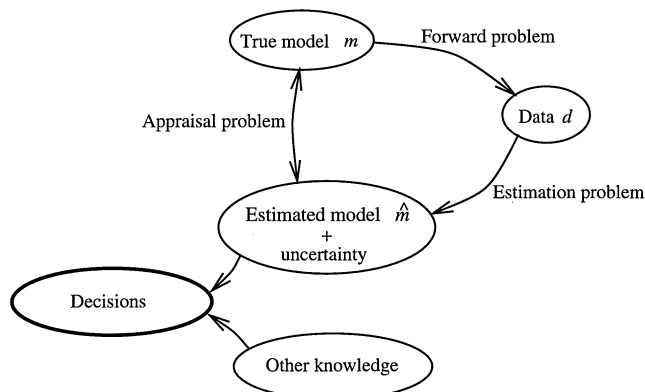


FIG. 3. The inverse problem as part of a decision-making process.

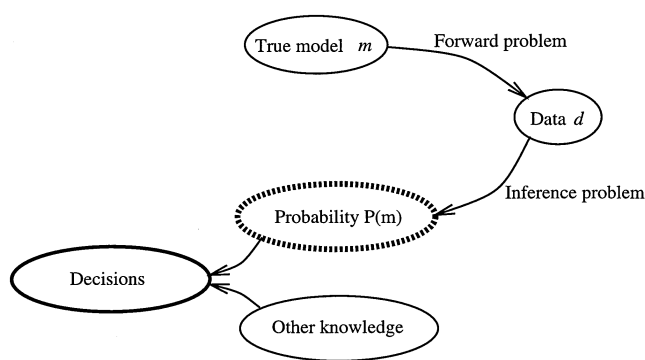


FIG. 4. The inverse problem as an inference problem.

simplest approach is to compute the mean and variance of each model parameter, and one can visualize this information relatively easily. However, as we have seen, the patterns in the model are much more interesting than the estimates of one model parameter. Assessing the robustness of certain patterns in the model is much more difficult, especially since this entails the use of the correlation between different model parameters. In a high-dimensional model space it is extremely difficult to characterize and interpret the correlations of the model parameters. In order to interrogate the resulting probability density function of the model in a meaningful way, research is needed into cluster and feature analysis of (possibly multimodal) probability density functions of many degrees of freedom, as well as the development of an interface between the exploration of these high-dimensional functions and the decision-making process.

It will be clear from this that in order to treat inverse problems in ways that are different from current practice requires significant theoretical and numerical advances. However, clear strategies for the optimal use of inverse problems in the process of decision making should also be an important item on the agenda for researchers in inverse problems.

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