1. Typically what we do with Einstein's equations is take a given distribution of sources, i.e. the energy-momentum tensor \( T_{\mu\nu} \), and then solve for the resulting metric \( g_{\mu\nu} \). However we can turn this around and ask, "given a particular geometry, i.e. a metric, what would be the required energy-momentum tensor to satisfy Einstein's equation?" This is a much easier task!

To see this consider the following metric:
\[
\begin{align*}
\text{ds}^2 &= -\cosh^2(\psi)\text{d}t^2 + \text{d}\psi^2 + \sinh^2(\psi)\text{d}x^2 + \sinh^2(\psi)\sin^2(\chi)\text{d}\delta^2 + \text{d}\eta^2 \\
&\quad + \sin^2(\eta)\text{d}\theta^2 + \sin^2(\eta)\sin^2(\theta)\text{d}\phi^2 + \sin^2(\eta)\sin^2(\theta)\sin^2(\phi)\text{d}\beta^2
\end{align*}
\]

Calculate the energy-momentum tensor that would create this geometry. Also, calculate the trace of this energy-momentum tensor.

See Mathematica notebook for solution.

2. Repeat our argument in class to arrive at the Schwarzschild solution, but now in 5D. Base your construction on SO(4) invariance (so the space will be foliated by three-spheres in analogy to the case we did in class). You only need to find the radial dependence of the metric. Do not worry about identifying the constant in your final expression. It will help to know that the line element for a three-sphere is given by
\[
\text{d}\Omega^2 = \text{d}x^2 + \sin^2(x)\text{d}\theta^2 + \sin^2(\theta)\text{d}\phi^2
\]

For the \( 4+1 \) dimensional Schwarzschild solution based on an \( S^3 \)-foliation, we know that spherical symmetry and coordinate redefinitions can be used to bring the metric to the form:
\[
\text{d}s^2 = -e^{-\Delta c(r,t)}\text{d}t^2 + e^{\Delta c(r,t)}\text{d}r^2 + r^2\text{d}\Omega^2
\]

where \( \text{d}\Omega^2 = \text{d}x^2 + \sin^2(x)\text{d}\theta^2 + \sin^2(\theta)\text{d}\phi^2 \), \( \chi, \theta \in [0, \pi] \), \( \phi \in [0, 2\pi] \).

Then using the vacuum form of Einstein's equation \( R_{\mu\nu} = 0 \):
\[
\begin{align*}
R_{tr} &= \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial G}{\partial r} = 0 \Rightarrow \Delta c(r,t) = \Delta c(r) \\
R_{44} &= e^{-2\Delta} \left[ -2 + 2\Delta - r\Delta' + r\Delta'' \right] = 0 \quad \text{where} \quad \Delta' = \frac{\partial}{\partial r} \Delta \quad \Delta'' = \frac{\partial^2}{\partial r^2} \Delta
\end{align*}
\]

\[
\begin{align*}
\partial_t R_{44} &= e^{-2\Delta} \left[ -\frac{\partial^2}{\partial t \partial r} \Delta \right] = 0 \Rightarrow \Delta (r,t) = \Delta (r) + q(t) \\
\text{redefining} \ t \ \text{can absorb this factor}
\end{align*}
\]
Then:
\[ ds^2 = -e^{-2\alpha(r)} dt^2 + e^{2\alpha(r)} dr^2 + r^2 d\Omega^2 \]

Now considering:
\[ R_{tt} = \frac{1}{r} e^{2\alpha - 2\beta} \left[ r \kappa'' + 3 \kappa' - \alpha' \beta' + \alpha'' \right] = 0 \]
\[ R_{rr} = -\kappa' + \frac{1}{r} 3 \beta' + \kappa' \beta' - \alpha'' = 0 \]

We can form:
\[ e^{-2\alpha + 2\beta} \left( R_{tt} + R_{rr} \right) = 0 = 3 \kappa' + 3 \beta' \Rightarrow \kappa(r) = -\beta(r) + C \]

So that:
\[ ds^2 = -e^{-2\beta(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \]

Finally:
\[ R_{tt} = e^{-2\beta} \left[ -2 + e^{2\beta} + 2\beta' r \right] = 0 \Rightarrow e^{-2\beta} e^{\beta} r = 1 \]

or
\[ -2 \beta + \frac{1}{r} \beta' \]

or
\[ \frac{1}{2} \frac{d}{dr} \left( r e^{-2\beta} \right) = r \]

So in the end:
\[ ds^2 = -(1 + \frac{c}{r^2}) dt^2 + (1 + \frac{c}{r^2}) dr^2 + r^2 d\Omega^2 \]

Note \( \frac{1}{r^2} \) instead of \( \frac{1}{r} \) as in \( 4D \).
3. Consider Einstein’s equations in a vacuum, but with cosmological constant $\Lambda$ such that

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$ 

a) Solve for the most general spherically symmetric metric that reduces to the Schwarzchild metric when $\Lambda \to 0$.

b) For the metric you derived, construct the effective radial potential for geodesic motion and plot the potential for massive particles with $L = 0$ for the three cases $\Lambda > 0, \Lambda = 0, \Lambda < 0$.

Note: These are values of the cosmological constant, not the angular momentum.

\[
\begin{align*}
\text{a)} \quad G_{\mu\nu} + \Lambda g_{\mu\nu} &= 0 \quad \text{or} \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \\
\text{Tracing both sides:} \quad R - \frac{1}{2} R + 4\pi \Lambda &= 0 \\
R &= 4\pi \Lambda \\
\text{Then our field equation becomes:} \quad R_{\mu\nu} = \Lambda g_{\mu\nu}
\end{align*}
\]

To find the most general spherically symmetric solution we can follow all of the steps we used to derive the Schwarzchild solution up until when we used the field equation ($R_{\mu\nu} = 0$ in that case).

Thus we start with:

$$ds^2 = -e^{2\phi(r)} dt^2 + e^{2\phi(r)} dr^2 + r^2 d\Omega^2$$

Now we need the components of $R_{\mu\nu}$, but since the metric so far is the same as the Schwarzchild case, they will be the same. We only modify the r.h.s. of EE.
Then:
\[ R_{tr} = \frac{1}{r} \frac{\partial \Phi}{\partial t} = 0 \quad \text{(since } \Phi_{tr} = 0) \quad \Rightarrow \quad \beta_{t}(r, t) = \beta(r) \]
\[ R_{oo} = e^{-2\phi} \left[ r \left( \frac{2}{r} - \frac{2}{r} \right) - 1 \right] + 1 = \Phi_{oo} = \Lambda r^{-1} \]

Again considering:
\[ \partial_t R_{oo} = -2 \frac{\partial \Phi}{\partial t} e^{-2\phi} \left[ r \left( \frac{2}{r} - \frac{2}{r} \right) - 1 \right] + e^{-2\phi} \left[ \frac{\partial}{\partial t} \left( \frac{\partial \Phi}{\partial t} \right) - \frac{\partial \phi}{\partial t} \phi \right] \]
\[ = \partial_t (\Lambda r^{-1}) = 0 \]
So:
\[ \frac{\partial \phi}{\partial t} = 0 \quad \Rightarrow \quad \alpha (r, t) = f(r) + g(t) \]

Then:
\[ ds^2 = -e^{-2f(r)} dt^2 + e^{2g(t)} dr^2 + r^2 d\Omega^2 \]

Defining:
\[ t' = \int e^{g(t)} dt \quad \Rightarrow \quad dt' = e^{g(t)} dt \]
\[ ds^2 = -e^{-2f(t')} dt'^2 + e^{2g(t')} + 1 d\Omega^2 \quad \text{still same as Schwarzschild!} \]

Then:
\[ R_{tt} = \frac{2}{r} \left[ \frac{\partial f}{\partial r} \right]^2 - \frac{\partial f}{\partial r} \frac{\partial g}{\partial r} + \frac{1}{r} \frac{\partial f}{\partial r} \] \[ = \Lambda \Phi_{tt} = -\Lambda e^{-2f} \]
\[ R_{rr} = -\frac{\partial^2 f}{\partial r^2} - \frac{\partial f}{\partial r} \frac{\partial g}{\partial r} + \frac{1}{r} \frac{\partial f}{\partial r} = \Lambda \Phi_{rr} = \Lambda e^{-2f} \]

Consider:
\[ -2 \frac{\partial f}{\partial r} \frac{\partial \Phi}{\partial r} \rightarrow \frac{1}{r} \left[ \frac{\partial f}{\partial r} + \frac{\partial g}{\partial r} \right] = 0 \]
\[ \Rightarrow \quad f(r) = -\beta(r) + C \]
So far:
\[ ds^2 = -e^{-2\phi(r)} \left( e^{2\phi(r)} dt^2 + dr^2 + r^2 d\Omega^2 \right) \]

Still same as Schwarzschild.

Finally:
\[ R_{\theta\theta} = e^{2\phi} ( -2r \frac{d\phi}{dr} - 1 ) + 1 = \Lambda g_{\theta\theta} = \Lambda r^2 \]

or
\[ \frac{2}{\phi}(r e^{2\phi}) = 1 - \Lambda r^2 \Rightarrow r e^{2\phi} = \int \left[ 1 - \Lambda r^2 \right] dr \]

\[ = r - \frac{1}{3} \Lambda r^3 + \xi \]

\[ e^{2\phi} = 1 - \frac{1}{3} \Lambda r^2 + \xi \frac{r}{r} = e^{-2\phi} \]

\[ ds^2 = -\left( 1 + \frac{2\phi}{r} - \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 + \frac{2\phi}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 d\Omega^2 \]

To agree with Schwarzschild when \( \Lambda \to 0 \) we need \( \xi = -2\phi \)

\[ ds^2 = -\left( 1 - \frac{2\phi}{r} - \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 - \frac{2\phi}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 d\Omega^2 \]

b) First note that the metric has no \( t \) or \( \phi \) dependence.

Thus we immediately know \( 2 \) Killing vectors:
\[ K_\theta = (1, 0, 0, 0) \Rightarrow K_\theta = \left( -\left[ 1 - \frac{2\phi}{r} - \frac{\Lambda r^2}{3} \right], 0, 0, 0 \right) \]

\[ K_\phi = (0, 0, 0, 1) \Rightarrow K_\phi = \left( 0, 0, 0, r^2 \sin^2 \theta \right) = (0, 0, 0, r^2) \]

= \( r^2 \) if we set \( \theta = \frac{\pi}{2} \).
Then we have the conserved quantities:

1) $K = \frac{d^2 x^4}{d\lambda^2} = (1 - \frac{2m}{r} - \frac{\Lambda r^4}{3}) \frac{dt}{d\lambda}$

2) $R = \frac{d^2 x^1}{d\lambda^2} = \frac{d^2 x^0}{d\lambda^2}$

Recall that for geodesics, $\lambda = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$ is conserved.

3) $\lambda = (1 - \frac{16\pi}{r} - \frac{4\pi^2}{3}) (\frac{dt}{d\lambda})^2 - (1 - \frac{16\pi}{r} - \frac{4\pi^2}{3}) (\frac{d\phi}{d\lambda})^2$

where $\lambda = 0$ for $\Lambda = 0$ and $\lambda = 1$ for $\Lambda > 0$.

Combining these three expressions (1-3), we have:

$(1 - \frac{16\pi}{r} - \frac{4\pi^2}{3}) \lambda = E^2 - (\frac{d\phi}{d\lambda})^2 - (1 - \frac{16\pi}{r} - \frac{4\pi^2}{3}) \frac{L^2}{r^2}$

or

$E^2 = \frac{1}{\lambda} (\frac{d\phi}{d\lambda})^2 + \frac{1}{\lambda} \lambda - \frac{G\Lambda}{r} + \frac{L^2}{2r^2} - \frac{GmL^2}{r^3} - \frac{\Lambda r^4}{6} - \frac{\Lambda L^2}{6}$

$V_{\text{eff}}(r)$

For $\Lambda > 0$ and $\lambda = 1$,

$\lambda = 0$ and taking $G\Lambda = 1$ and $L = 0$ for purely radial motion we have:

$V_{\text{eff}}(r) = \frac{1}{\Lambda} - \frac{1}{r} - \frac{\Lambda r^2}{6}$