General Relativity HW11 Problems

1. Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional spacetime, equivalently, a cylindrical configuration in (3+1)-dimensions with perfect rotational symmetry.
   a) Show that the vacuum solution can be written as
      \[ ds^2 = -dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 \]
      where \( M \) is constant and \( \theta \in [0, 2\pi) \).
   b) Derive the analogue of the Tolman-Oppenhiemer-Volkoff equation for (2+1)-dimensions.
   c) Solve the (2+1)-dimensional TOV equation for a constant density star. Find \( p(r) \) and solve for the metric.

2. Once across the event horizon of a Schwarzschild black hole of mass \( M \), what is the longest proper time an observer can spend before reaching the singularity?

3. Consider the spacetime specified by the line element
   \[ ds^2 = -(1 - \frac{GM}{r})^2 dt^2 + (1 - \frac{GM}{r})^{-2} dr^2 + r^2 d\Omega^2 \]
   a) Find a transformation to Eddington-Finkelstein-like coordinates \((\nu, r, \theta, \phi)\) such that \( g_{rr} = 0 \) and show that the geometry is not singular at \( r = GM \).
   b) Sketch a plot analogous to our picture in class (EF for Schwarzschild) of the light cones in this geometry.