1. Imagine a particle following a path through spacetime given by $x^\mu(\tau) = \left( \tau^2 + \tau, \tau^2 + \frac{4}{3} \tau^3, -10 \right)$.

   a) Compute the four-velocity of the particle as it passes through the point $x^\mu = (20, 16, \frac{32}{3}, -10)$.

   First note that the point under consideration corresponds to $\tau = \eta$.

   Then we want $U^\mu(\tau = \eta) = \frac{dx^\mu}{d\tau} \bigg|_{\tau = \eta}$

   $U^\mu(\tau) = \left( 2\tau + 1, 2\tau, 2\tau^2, 0 \right)$

   Thus:

   $U^\mu(\tau = \eta) = \left( 9, 8, \eta, 0 \right)$

   b) For the function $f(t, x, y, z) = -t^2 + x^2 + y^2 - yz$, calculate the rate of change of this function along the path, i.e. $\frac{\partial f}{\partial \tau}$, at the point $x^\mu = (20, 16, \frac{32}{3}, -10)$.

   Hint: You will need to break up the directional derivative into two terms using $\frac{\partial x^\mu}{\partial x}$ in various places so that can use your result for the four-velocity.

   To evaluate $\frac{\partial f}{\partial \tau}$ consider $\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\tau} = \frac{\partial f}{\partial x^\mu} \frac{U^\mu}{d\tau}$ from part (a)

   First:

   $\frac{\partial f}{\partial x^\mu} = \left( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$

   $= \left( -2t, 2x, 2y-z, -1 \right)$

   Then:

   $\frac{\partial f}{\partial x^\mu} \frac{U^\mu}{d\tau} = \left( -2t, 2x, 2y-z, -1 \right) \frac{\left( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)}{\left( \frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial \tau} \right)} = -2t \left( \frac{\partial f}{\partial \tau} \right) + 4x \frac{\partial f}{\partial \tau} + 4yz^2 - 2z \frac{\partial f}{\partial \tau}$

   Then at $\tau = \eta$ or $\tau = 20$, $x = 16$, $y = \frac{32}{3}$, $z = -10$ we have

   $\frac{\partial f}{\partial \tau} \bigg|_{\tau = \eta} = -\eta \left( 9 \right) + 256 + 85.33 + 40 = 213.33$
1. \( T_{\text{rest}} = \begin{pmatrix} \rho & 0 \\ 0 & \rho & 0 \end{pmatrix} \)

a) To boost along the diagonal we first rotate \( \text{CCW} \) in \( x-y \) by \( \frac{\pi}{4} \):

Then boost by \( v \) along \( +x'' \):

Finally rotate back:

We can write this as:

\[
T^{m'} = \Lambda_{m''}^{m'} \Lambda_{m'}^{m''} \Lambda_{m''}^{m'} T^{m} \Lambda_{m''}^{m} \Lambda_{m}^{m'} v'' \Lambda_{v''}^{v'}\]

Recall that as matrices we have:

If \( \Lambda_{m'}^{m} \rightarrow \Lambda \)

then \( \Lambda_{m''}^{m'} \rightarrow \Lambda^T \)

Before we begin explicit calculation we realize that since \( T^{m} \) is isotropic in \( x, y, z \) that any rotation without boosting will not change its form. So we may write:

\[ \Lambda_{m''}^{m'} \Lambda_{m'}^{m''} v'' = \begin{pmatrix} \rho & 0 \\ 0 & \rho & 0 \end{pmatrix} \]

Then we have:

\[
T^{m'} = \Lambda_{m''}^{m'} \left( \begin{array}{ccc} \cosh \phi & -\sinh \phi & 0 \\ -\sinh \phi & \cosh \phi & 0 \\ 0 & 0 & 1 \end{array} \right) \begin{pmatrix} \rho & 0 \\ 0 & \rho & 0 \end{pmatrix} \left( \begin{array}{ccc} \cosh \phi & -\sinh \phi & 0 \\ -\sinh \phi & \cosh \phi & 0 \\ 0 & 0 & 1 \end{array} \right) \Lambda_{v''}^{v'}\]

\[= \Lambda_{m''}^{m'} \left( \begin{array}{ccc} \cosh \phi + p \sinh \phi & -p \cosh \phi & 0 \\ -p \sinh \phi & \cosh \phi & 0 \\ 0 & 0 & 1 \end{array} \right) \Lambda_{m'}^{m''} v'' \]

\[= \Lambda_{m''}^{m'} \left( \begin{array}{ccc} \gamma^2 (\rho + v^2 p) & -\gamma^2 v (\rho + p) & 0 \\ -\gamma^2 v (\rho + p) & \gamma^2 (\rho^2 + p^2) & 0 \\ 0 & 0 & 1 \end{array} \right) \Lambda_{m'}^{m''} v'' \]

Having used \( \cosh \phi = \gamma \), \( \sinh \phi = \gamma v \)

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma^2 (\rho + v^2 p) & -\gamma^2 v (\rho + p) & 0 \\ -\gamma^2 v (\rho + p) & \gamma^2 (\rho^2 + p^2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}
\]

\[
= \begin{pmatrix} \gamma^2 (\rho + v^2 p) & -\gamma^2 v (\rho + p) & 0 \\ -\gamma^2 v (\rho + p) & \gamma^2 (\rho^2 + p^2) & 0 \\ -\gamma^2 (\rho^2 + p^2) & \gamma^2 (\rho^2 + p^2) & 0 \end{pmatrix}
\]
b) The expression for a perfect fluid $\mathbf{T}^{\mu\nu}$ valid in any frame is: $\mathbf{T}^{\mu\nu} = (\rho + p) U^\mu U^\nu + p n^\mu n^\nu$

where $U^\mu$ defines the transformation from the rest frame to the frame in which we are observing.

The relationship between these frames is $v^\mu = \frac{v^\mu}{\sqrt{1 - \text{v} \cdot \text{v}}}$ so we have $v_x = \frac{v}{\sqrt{1 - \text{v} \cdot \text{v}}}$, $v_y = \frac{v}{\sqrt{1 - \text{v} \cdot \text{v}}}$, $v_z = 0$

and thus $U^\mu = (1 - \frac{v \cdot v}{\sqrt{1 - \text{v} \cdot \text{v}}}, \frac{v \cdot n}{\sqrt{1 - \text{v} \cdot \text{v}}}, 0, 0)$.

In this expression remember $\gamma = \frac{1}{\sqrt{1 - \text{v} \cdot \text{v}}}$ where this $\gamma$ is the same as in $v_x, v_y$ above (it is the velocity connecting the frames) and note that the components of the 3-velocity of the fluid are negative since in the $S'$ frame the fluid is moving $\checkmark$ (it was at rest in $S$).

Then:

$$\mathbf{T}^{\mu\nu} = (\rho + p) U^\mu U^\nu + p n^\mu n^\nu$$

$$= (\rho + p) \begin{pmatrix} U^0 U^0 & U^0 U^1 & U^0 U^2 & U^0 U^3 \\ U^1 U^0 & U^1 U^1 & U^1 U^2 & U^1 U^3 \\ U^2 U^0 & U^2 U^1 & U^2 U^2 & U^2 U^3 \\ U^3 U^0 & U^3 U^1 & U^3 U^2 & U^3 U^3 \end{pmatrix} + p \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (\rho + p) \begin{pmatrix} \gamma^2 & -\gamma^2 v_x & -\gamma^2 v_y & -\gamma^2 v_z \\ -\gamma^2 v_x & \gamma^2 & 0 & 0 \\ -\gamma^2 v_y & 0 & \gamma^2 & 0 \\ -\gamma^2 v_z & 0 & 0 & \gamma^2 \end{pmatrix} + \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2(\rho + p) & -\gamma^2 v_x(p + p) & -\gamma^2 v_y(p + p) & -\gamma^2 v_z(p + p) \\ -\gamma^2 v_x(p + p) & \gamma^2(p + p) & 0 & 0 \\ -\gamma^2 v_y(p + p) & 0 & \gamma^2(p + p) & 0 \\ -\gamma^2 v_z(p + p) & 0 & 0 & \gamma^2(p + p) \end{pmatrix}$$

Comparing to the result of (a) we note a few terms that look different:

$-T^{00'} = \gamma^2 \rho + (\gamma^2 - 1)p = \gamma^2 \rho + \gamma^2 v^2 p = \gamma^2(\rho + v^2 p)$ having used $1 - v^2 = \gamma^2 v^2$

$T^{11'} = \gamma^2 v^2 \rho/2 + (1 + \gamma^2 v^2 p) = \gamma^2 v^2 \rho/2 + \frac{1}{2}(\gamma^2 - 1)p = \gamma^2 v^2 \rho/2 + \frac{1}{2}(\gamma^2 - 1)p$

$= \gamma^2(v^2 \rho + p)/2 + \frac{1}{2}p$

$T^{22'} = T^{33'} = \gamma^2 v^2 (p + p)/2 = \gamma^2 v^2 \rho/2 + \frac{1}{2}(\gamma^2 - 1)p = \gamma^2(v^2 \rho + p)/2 - \frac{1}{2}p$

So the results from (a) and (b) agree.
3. The perfect fluid energy-momentum tensor is the fluid rest frame $T_{\mu\nu}^\text{rest} = \text{diag}(\rho, p, p, p)$ becomes for a vacuum w/ equation of state $\rho = -p$:

$$T_{\text{vac, rest}}^{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p \end{pmatrix} = \rho \eta^{\mu\nu} \quad \text{where } \eta^{\mu\nu} \text{ is the inverse of the Minkowski metric.}$$

But we know that $\eta^{\mu\nu}$ (and $\eta_{\mu\nu}$) are invariant under Lorentz transformations (which you can check explicitly) so we can immediately conclude that $T_{\text{vac}}^{\mu\nu}$ takes the same form in all inertial frames. This is not surprising since it doesn’t mean much to be “at rest” w/ respect to the vacuum.

One could say that all frames are at rest w/ respect to the vacuum, but the important point is that no single frame is distinguished.