General Relativity HW6 Problems

1. Argue that naively adding a finite speed of propagation to Newtonian gravity will result in stable orbits becoming unstable. A simple scenario to consider is two equal mass objects in a circular orbit around a common point under their mutual gravitational interaction.

2. Suppose that we lived in a world with only two forces: gravity and the electromagnetic force. Also suppose that every bit of matter in this world was “extremal”, i.e. the electromagnetic charge of any particle is always equal to its mass (with a universal constant to get the dimensions correct). In this context, does it make sense to single out gravity as providing the “curvature of spacetime”, or could we instead use electromagnetism, or perhaps both?

3. When we constructed the atlas for a circle in class, we started with two labels θ, φ that ran from [0,2π). It was the closed end of these intervals that prevented us from using just one of them as a single chart covering the space. Why would the following not have helped:
   a) Use one label with (0,2π).
   b) Use one label with (0,2π + ε).

4. Given that a circle requires at least two charts to form an atlas, it might be surprising that the surface of an infinite cylinder can be covered with an atlas consisting of only one chart. Construct such an atlas for the cylinder. Remember, this is now a 2D space.

   For the next two questions, recall that the interval \( ds^2 \) for the coordinate displacements \( dx^\mu = (dx^1, dx^2, ...) \) can be obtained from the metric \( g_{\mu\nu} \) by \( ds^2 = dx^\mu g_{\mu\nu} dx^\nu \).

5. Consider \( \mathbb{R}^3 \) as a manifold with the flat Euclidean metric, and coordinates \( \{x, y, z\} \). Introduce spherical polar coordinates \( \{r, \theta, \phi\} \) related to \( \{x, y, z\} \) by
   \[
   x = r \sin \theta \cos \phi \\
   y = r \sin \theta \sin \phi \\
   z = r \cos \theta
   \]
   so that the metric takes the form
   \[
   ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
   \]
   a) If a particle moves along the parameterized curve given by
      \[
      x(\lambda) = \cos \lambda \quad y(\lambda) = \sin \lambda \quad z(\lambda) = \lambda
      \]
   express the path of the curve in the \( \{r, \theta, \phi\} \) coordinate system.
   b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

6. Prolate spheroidal coordinates are related to the usual Cartesian coordinates \( \{x, y, z\} \) of Euclidean three-space by
   \[
   x = \sinh \chi \sin \theta \cos \phi \\
   y = \sinh \chi \sin \theta \sin \phi \\
   z = \cosh \chi \cos \theta
   \]
   Restrict your attention to the \( y = 0 \) plane and answer the following:
   a) What is the coordinate transformation matrix \( \frac{\partial x^\mu}{\partial x^\nu} \) relating \( \{x, y\} \) to \( \{\chi, \theta\} \).
   b) What does the invariant interval \( ds^2 \) look like in prolate spheroidal coordinates?