You know the drill!

1. For a 2-sphere with coordinates \((\theta, \phi)\), write down the equations for parallel transport of a vector along a line of constant **longitude**. Then parallel transport the vector with components \(V^\mu = (1, 0)\) once around the line and write down the result. You may use any results from your homework without deriving them again.

For \(S^2: \Theta (\theta, \phi)\) we have: \(\Gamma^\phi_\theta = \Gamma^\theta_\phi = \cot \theta\), \(\Gamma^\phi_\phi = -\sin \phi \cos \theta\).

For \(\parallel\text{-transport of } V^\mu\) we require: \(\frac{dV^\mu}{d\lambda} + \Gamma^\mu_\nu \frac{dx^\nu}{d\lambda} V^\nu = 0\).

For a line of constant longitude: \(x^\mu (\lambda) = (\lambda, \phi_0) \Rightarrow \frac{\partial \theta}{d\lambda} = 1, \frac{\partial \phi}{d\lambda} = 0\).

Then:

\(\Theta: \frac{dV^\theta}{d\lambda} + \Gamma^\theta_\phi \frac{d\phi}{d\lambda} V^\phi = \frac{dV^\theta}{d\lambda} = 0 \Rightarrow V^\theta = \text{constant}\)

\(\Phi: \frac{dV^\phi}{d\lambda} + \Gamma^\phi_\theta \frac{d\theta}{d\lambda} V^\theta + \Gamma^\phi_\phi \frac{d\phi}{d\lambda} V^\phi = \frac{dV^\phi}{d\lambda} + \cot \phi \frac{dV^\phi}{d\lambda} = 0\)

\(\frac{dV^\phi}{d\lambda} = -\cot \phi \frac{dV^\phi}{d\lambda} \Rightarrow V^\phi = A e^{-\cot \phi \lambda} \Rightarrow V^\phi (\lambda = 0) = 0 \Rightarrow A = 0\)

So, \(V^\mu_{\parallel} = (1, 0)\).

Alternatively:

\(V^\mu = (1, 0)\) is tangent to the curve \(x^\mu (\lambda) = (\lambda, \phi_0)\) and \(x^\mu (\lambda)\) is a geodesic, so the tangent vector should not change as it is \(\parallel\text{-transported along } x^\mu (\lambda)\).
2. Consider the upper-half plane model of the hyperbolic plane \( H = \{(x, y) \in \mathbb{R}^2 | y > 0\} \) with line element \( ds^2 = \frac{dx^2 + dy^2}{y^2} \). Find the form of the divergence operator on a vector function \( V^u (x, y) \) in the coordinate basis.

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\begin{align*}
\nabla_a V^a &= \partial_a V^a + \Gamma^a_{mn} V^m \partial_n V^a = \partial_1 V^1 + \partial_2 V^2 + \Gamma^1_{12} V^1 + \Gamma^1_{21} V^2 + \Gamma^2_{12} V^1 + \Gamma^2_{21} V^2 \\
\text{Note: } & 
\begin{align*}
\partial_a (y^{-2} \partial_a) &\text{ so we can use the results of problem 3.} \\
\Gamma^1_{11} &= \partial_1 (\ln \sqrt{y^2 + 1}) = \partial_x (\ln \sqrt{1 + y^2}) = 0 \\
\Gamma^2_{11} &= \partial_1 (\ln \sqrt{y^2 + 1}) = \partial_x (\ln \sqrt{1 + y^2}) = 0 \\
\Gamma^1_{12} &= \partial_2 (\ln \sqrt{y^2 + 1}) = \partial_y (\ln \sqrt{y^2 + 1}) = \frac{-2y}{y^2 + 1} = -\frac{1}{y} \\
\Gamma^2_{12} &= \partial_2 (\ln \sqrt{y^2 + 1}) = \partial_y (\ln \sqrt{y^2 + 1}) = -\frac{1}{y} \\
\text{Then: } & \nabla_a V^a = \partial_1 V^1 + \partial_2 V^2 - \frac{1}{y} V^1
\end{align*}
\end{align*}
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