1. Find explicit expressions for all of the Killing vectors $K^\mu$ for 1+3D Minkowski space $\mathbb{M}^4$. Be careful to recall that what appears in Killing’s equation are the dual Killing vectors!

\[ \text{In } \mathbb{M}^4 \text{ with } (t, x, y, z), \quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \]

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

All $\Gamma^\mu_\nu = 0$.

Then $\nabla_\mu K_\nu$ becomes:

\[ \partial_t K_t = 0 \quad \partial_t K_x + \partial_x K_t = 0 \quad \partial_t K_y + \partial_y K_t = 0 \]

\[ \partial_x K_x = 0 \quad \partial_x K_y + \partial_y K_x = 0 \quad \partial_x K_z + \partial_z K_x = 0 \]

\[ \partial_y K_y = 0 \quad \partial_y K_z + \partial_z K_y = 0 \quad \partial_y K_\gamma + \partial_\gamma K_y = 0 \]

\[ \partial_z K_z = 0 \quad \partial_z K_\gamma + \partial_\gamma K_z = 0 \]

All 10 of these must be satisfied by any solution.

Since $\mathbb{M}^4$ is maximally symmetric we expect $\frac{1}{4} \binom{4 + 4}{4} = 10$ solutions.

For each solution $K_\mu$ we find the corresponding Killing vector $K^\mu = g^{\mu\nu} K_\nu$ and the conserved dual momentum $P_\mu = K^\mu P_\mu$ where $P_\mu = (-E, P_x, P_y, P_z)$.

4 easy ones are:

\[ K_\mu = (1, 0, 0, 0) \Rightarrow K^\mu = (1, 0, 0, 0) \Rightarrow K^\mu P_\mu = E \]

\[ K_\mu = (0, 1, 0, 0) \Rightarrow K^\mu = (0, 1, 0, 0) \Rightarrow K^\mu P_\mu = P_x \]

\[ K_\mu = (0, 0, 1, 0) \Rightarrow K^\mu = (0, 0, 1, 0) \Rightarrow K^\mu P_\mu = P_y \]

\[ K_\mu = (0, 0, 0, 1) \Rightarrow K^\mu = (0, 0, 0, 1) \Rightarrow K^\mu P_\mu = P_z \]

Expected since $g_{\mu\nu}$ does not depend on $t, x, y, z$!

3 less obvious ones are:

\[ K_\mu = (1, -1, 0, 0) \Rightarrow K^\mu = (1, -1, 0, 0) \Rightarrow K^\mu P_\mu = -y P_x + x P_y = L_z \]

\[ K_\mu = (0, 0, 1, 0) \Rightarrow K^\mu = (0, 0, 1, 0) \Rightarrow K^\mu P_\mu = 2 P_x - x P_z = L_y \]

\[ K_\mu = (0, 0, 0, 1) \Rightarrow K^\mu = (0, 0, 0, 1) \Rightarrow K^\mu P_\mu = y P_x - z P_y = L_x \]

These 3 are expected since $\mathbb{M}^4$ is spatially isotropic (rotationally invariant).
Finally, 3 less intuitive solutions are:

\[ K_\alpha = (-x, t, 0, 0) \Rightarrow K^\gamma = (x, t, 0, 0) \Rightarrow K^\gamma P_\alpha = xE + tP_x \quad \text{Strange, but true!} \]

\[ K_\alpha = (\gamma, 0, t, 0) \Rightarrow K^\gamma = (\gamma, 0, t, 0) \Rightarrow K^\gamma P_\alpha = \gamma E + tP_\gamma \]

\[ K_\alpha = (-2, 0, 0, t) \Rightarrow K^\gamma = (2, 0, 0, t) \Rightarrow K^\gamma P_\alpha = 2E + tP_2 \]

Problems 2, 3 and 4, see Mathematica notebook for solutions.