1. Consider a unit 2-sphere with coordinates \((\theta, \phi)\) and metric \(ds^2 = d\theta^2 + \sin^2 \theta d\phi^2\).
   a) Take a vector with components \(V^\mu = (1,0)\) and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector as a function of the polar angle \(\theta\) of the circle of constant latitude.
   b) Show that lines of constant longitude \((\phi = \text{constant})\) are geodesics, and that the only line of constant latitude \((\theta = \text{constant})\) that is a geodesic is the equator \((\theta = \frac{\pi}{2})\).

2. Show that extremizing the functional \(\int ds\) for the space \(\mathbb{R}^2\) in polar coordinates \((r, \theta)\) leads to a set of differential equations for \(r(\lambda)\) and \(\theta(\lambda)\) that are the same as the geodesic equations.

3. Okay, just to make sure we all ended up on the same page at the end of our discussion of geodesics and the twin paradox, consider the set of time-like paths in \(\mathbb{M}^2\) (2D Minkowski space) that connect two points at the same spatial position and separated in time by some amount \(\Delta t\) (according to coordinates adapted to some inertial observer). We argued in class that the geodesic path between these points is the spacetime path of maximal length. While you can make an arbitrarily short geodesic path, there is a limit to how short time-like geodesics can be in this scenario. Argue that the set of all time-like paths (geodesic or otherwise) between the two points is bounded from below by a minimum length. Do this both geometrically (draw pictures) and from what you know about the invariant interval and time-like and light-like paths. What is different about this question compared to the case in \(\mathbb{R}^2\) where we do not have both upper and lower bounds on paths between two points?

For the following questions you may find it useful to use the Mathematica package G.R.E.A.T.
I will be posting solutions as Mathematica notebooks using the package. You are free to use other resources if you like, but my familiarity is limited to Mathematica.

4. The metric on a three-sphere in coordinates \(x^\mu = (\psi, \theta, \phi)\) can be written as:
   \[ ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \]
   Note: This not a ball in \(\mathbb{R}^3\), but rather a “surface” in \(\mathbb{R}^4\) defined by \(x^2 + y^2 + z^2 + w^2 = 1\).
   a. Calculate the Christoffel connection coefficients using whatever method you like.
   b. Calculate the Riemann tensor, Ricci tensor and Ricci scalar.
   c. Show that this space satisfies the expression \(R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})\)

5. This is your first project dealing with a space that is not quite trivial and whose metric you will have to figure out on your own. It’s time to think about the two-torus, i.e. what many of you might call a “doughnut”. A general n-torus can be defined as an n-dimensional rectangle (a patch of Euclidean space \(\mathbb{R}^n\)) with the opposite sides identified. So a 1D torus is a circle, a 2D torus is the usual “doughnut”, and in 3D and higher it gets hard to visualize. By this definition the torus it is intrinsically flat since it starts with a patch of \(\mathbb{R}^n\) (which is flat) and simply adds nontrivial identifications (periodicity of the edges).
However you may suspect that trying to build a two-torus that lives in 3D Euclidean space by this definition is a bit of a problem. Consider taking a piece of flat paper (a rectangular patch of Euclidean $\mathbb{R}^2$) and forming a torus. We can easily roll the paper into a cylinder without “stretching” it. But to finish the torus we have to “roll” the cylinder so that the two circles on the ends meet. But you cannot do this without stretching or shrinking the paper, i.e. changing the distance between points, i.e. changing the metric! Essentially, the two-torus as we have defined it (by identifications and hence intrinsically flat) cannot live in 3D Euclidean space (though it can live in 4D or higher).

To help make this clear, consider the 2D surface of a “doughnut” living in 3D. We won’t call this a torus anymore. Construct a set of good coordinates for this doughnut surface and determine the metric in those coordinates. Hint: One way to do this is to use the embedding map into 3D, $(x,y,z) = ((R_1 + R_2\sin v)\cos u, (R_1 + R_2\sin v)\sin u, R_2\cos v)$ where $(u,v)$ are the two coordinates on the doughnut. Then using the line element in 3D Euclidean space for $(x,y,z)$, you can convert this into a line element for $(u,v)$. **Hint:** Just plug in the embedding map above. Finally calculate the Riemann curvature tensor for your 2D metric, which will answer the question “Is the doughnut flat?” You will need the Mathematica G.R.E.A.T. package for this!

Lastly, to see that the flat two-torus can consistently live in 4D Euclidean space, consider the embedding map $(x,y,z,w) = (R_1\sin u, R_1\cos u, R_2\sin v, R_2\cos v)$. Construct the 2D metric for this torus and then compute the Riemann curvature tensor from it.