So far in our discussion of manifolds we know that we can construct an atlas of charts $\{(U, \phi_i)\}$ and use the maps $\phi_i : U_i \to \mathbb{R}^n$ to coordinatize $M$ (on each chart and then sew the patches together w/ transition functions, i.e. coord. changes).

One important thing to remember is that we cannot always cover $M$ w/ one chart, hence we cannot have one "global" coordinate system on $M$. If we can cover $M$ w/ one chart then we can (if we want) set up a global coordinate system.

Okay, so we use coordinate transformations to piece together patches, but what about a coordinate transformation for a global set of coordinates?

**Example:** $M = \mathbb{R}^n$

That is, we just use two charts covering $M$ but equipped w/ different maps!

Okay, coordinate changes... stuff said.
If we know how coordinate changes happen, can we say anything about derivatives w.r.t. coordinates?

There are 41 expressions here in 4D

A coordinate change gives: \( x'^{\mu} (x^{\nu}) \)

new as function of old

Consider the chain rule: \( \frac{d}{dx} \phi(q(x)) = \frac{dq}{dx} \frac{d\phi}{dq} \)

secretly \( \frac{d}{dx} (\phi \circ q) \)

Now bump it up: \( g_1 (x, y) \ g_2 (x, y) \)

\( \phi (g_1, g_2) \)

\[ \begin{align*}
\frac{\partial}{\partial x_1} \phi (q^{\lambda}) &= \frac{\partial \phi}{\partial q^1} \frac{\partial q^1}{\partial x_1} + \frac{\partial \phi}{\partial q^2} \frac{\partial q^2}{\partial x_1} \\
\frac{\partial}{\partial x_2} \phi (q^{\lambda}) &= \frac{\partial \phi}{\partial q^3} \frac{\partial q^3}{\partial x_2} + \frac{\partial \phi}{\partial q^4} \frac{\partial q^4}{\partial x_2}
\end{align*} \]

Now think of \( q \) as a coordinate change: \( g^{\mu'} (q^{\nu}) \) or \( x'^{\mu} (x^{\nu}) \)

Then: \( \frac{\partial}{\partial x_1} \phi (x'^{\mu}) = \frac{\partial \phi}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x_1} \Rightarrow \frac{\partial}{\partial x_1} = \frac{\partial x^{\mu'}}{\partial x_1} \frac{\partial}{\partial x'^{\mu'}} \)

\( \frac{\partial}{\partial x_2} \phi (x'^{\mu}) = \frac{\partial \phi}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x_2} \)

Transformation Law

for partial derivatives
What about more complicated objects like vectors, etc.? We know that these objects live in tangent and cotangent spaces, so let's get a formal definition of these. First off, they should be well-defined even without introducing coordinates (that is an option).

Begin w/ Manifold M
Point P ∈ M
Curve γ through P
γ(λ) : R → M (only gives position on curve)
λ ∈ R

Now add a C∞ map f : M → R
f(p) = αp (works for any PEM!)

This is not a chart! That is f o γ(λ)p = αp

In words: Changing λ changes p which changes α. So we can consider \[ \frac{df}{dλ} = \frac{∂f}{∂λ}(f o γ) \]
The directional derivative of f along λ (which is tangent to the curve γ).

If we now consider all curves through P parameterized by (λ, α, β, etc.) then the set of all directional derivatives for these curves forms a vector space \[ \mathbb{E} \] (\[ \frac{∂}{∂λ}, \frac{∂}{∂α}, \frac{∂}{∂β}, \text{etc.} \])

Recall that a vector space satisfies:

1) U + W = Y, closure under +
2) a U = Y, closure under scalar x
3) Associativity of +
4) Commutativity of +
5) Identity in +
6) Inverse in +
7) (a, b, i, etc.) rules for scalar x

So we have established a vector space of tangents to M at P, i.e. the tangent space at P!
Now that we have a tangent space, let's see if we can make use of the charts (and coordinates) to setup a basis and thus coordinate representations of vectors. Then we can ask how components transform under coordinate transformations.

Here comes the magic! \[
\frac{df}{d\lambda} = \frac{d}{d\lambda} \left( f \circ \phi^{-1} \circ \phi \circ \gamma \right)
\]

Since: \((f \circ \phi^{-1}) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) we can call it \(x^i(\lambda)\).

Since: \((f \circ \phi) : \mathbb{R}^m \rightarrow \mathbb{R}^n \) we can call it \(f(x^i)\).

Then: \[
\frac{df}{d\lambda} = \sum_{i=1}^{m} \frac{d}{d\lambda} f(x^i) = \sum_{i=1}^{m} \frac{d}{d\lambda} f_i \]

But if is arbitrary so: \[
\frac{df}{d\lambda} = \sum_{i=1}^{m} \frac{d}{d\lambda} f_i \quad \text{any directional derivative, i.e. tangent vector}
\]

We have established what is called a coordinate-adapted basis in the tangent space.
But wait... any vector \( \mathbf{v} \) must lie in the tangent space, and so must be expressible as \( \mathbf{v} = v^\alpha \partial_\alpha \).

It gets better because we know that \( \mathbf{v} \) is invariant under coordinate changes and \( \partial_\alpha \to \partial'_\alpha = \frac{\partial x^m}{\partial x'^\alpha} \partial_m \).

So we can infer that:
\[
\begin{align*}
\mathbf{v}^\alpha \partial_\alpha &= v'^\alpha \partial'_\alpha \\
&= v'^\alpha \frac{\partial x^m}{\partial x'^\alpha} \partial_m \\
&\downarrow \\
&= \mathbf{v} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\alpha} \partial_m
\end{align*}
\]

\( \therefore \mathbf{v}^\alpha \to v'^\alpha = \frac{\partial x^m}{\partial x^\alpha} \mathbf{v}^\alpha \) is transformation law for vector components.
Dual Vectors

We can now proceed much like what we did in \( \text{SR} \).

\[ w = w^m \hat{\epsilon}^m \quad \text{where} \quad \hat{\epsilon}^m \hat{\epsilon}^n = \delta^m_n \]

\[ \text{We know these are } dx \]

\[ \text{We will call these } dx^m \]

Then: \( dx^m \hat{\epsilon}^m = dx^\mu \hat{\epsilon}^\mu \Rightarrow dx^\mu = \frac{\partial x^\mu}{\partial x^m} dx^m \quad \text{Transformation law for dual basis vectors} \]

And: \( w_m \rightarrow w_\mu = \frac{\partial x^\mu}{\partial x^m} w_m \quad \text{Transformation law for dual vector components} \)

Tensors

No surprise here: \( T^\mu_{\alpha \beta} \rightarrow T^\mu'_{\alpha' \beta'} = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\beta} T^\nu_{\mu \beta} \)

\( \text{Transformation law for tensor components} \)