Some specifics about the general coordinate transformations.

\[
\frac{\delta x'}{\delta x''} \frac{\delta x''}{\delta x'} = \mathcal{M} \\
\mathcal{M}^{-1} = \mathcal{M}
\]

Example: \(x' = x' + y^2 \Rightarrow \frac{\partial x'}{\partial x} = 1, \frac{\partial x'}{\partial x'} = 1 \Rightarrow \frac{\partial x''}{\partial x'} = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = \mathcal{M}
\]
\(x'' = -x' + y^2 \Rightarrow \frac{\partial x''}{\partial x'} = -1, \frac{\partial x''}{\partial x''} = 1 \)

\[
\left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \]  \(\mathcal{M}^{-1} = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right) = \frac{\partial x'}{\partial x''}
\]

By golly, it all checks out!
So let's compare: Special Relativity \( V^\mu \rightarrow V'^\mu = \Lambda^\mu_\nu V^\nu \)
General Relativity \( V^\mu \rightarrow V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \)

This allows "local" or coordinate dependent transformations of \((t,x,y,z)\), i.e. \(\frac{\partial x'^\mu}{\partial x^\nu}(x^\nu)\)

The crucial difficulty we will encounter is that unlike the constant \(\Lambda^\mu_\nu\)'s, the \(\frac{\partial x'^\mu}{\partial x^\nu}(x^\nu)\) cannot in general be moved past derivatives! Why does this matter?

Consider the derivative of a tensor:

\[
\partial_\mu T^\nu \rightarrow \partial'_{\nu'} T'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^\nu} \partial_\mu \left( \frac{\partial x^\nu}{\partial x'^{\nu'}} T^\nu \right) = \frac{\partial x'^{\mu'}}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^{\nu'}} \partial_\mu T^\nu + T^\nu \frac{\partial x'^{\mu'}}{\partial x^\nu} \partial_\nu \left( \frac{\partial x^\nu}{\partial x'^{\nu'}} \right)
\]

This is all we should get if \(\partial_\mu T^\nu\) was a tensor, but...

So in contrast to SR, the derivative of a tensor is generally not itself a tensor. But we need derivatives for physics and we need tensor equations \(\Rightarrow\) We need a new derivative!
Lecture 13 - Metrics, Flatness and LICs

Recall that the metric provides a one-to-one correspondence between vectors and dual vectors; i.e. it raises or lowers indices while maintaining the underlying tensor.

\[ g_{\mu \nu} \bar{\nu}^\alpha = \bar{T}^\alpha_\alpha \] (in contrast to \( \bar{h}_{\alpha \nu} \bar{\nu}^\alpha = \bar{J}^\alpha_\alpha \))

In particular \( g^{\alpha \mu} g_{\nu \mu} = \delta^\alpha_\nu \) (or \( g^{\alpha \mu} = (g_{\nu \mu})^{-1} \)) sets it apart.

Contrast with \( T^\alpha_\mu \neq (T^\alpha_\alpha)^{-1} \)

Quite often we will express the metric by a line element:

\[ \mathbb{R}^3 \{ r, \theta, \phi \} \Rightarrow g_{\mu \nu} = \begin{pmatrix} 1 & 0 & 0 \\ r^2 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \Rightarrow ds^2 = dx^\mu g_{\mu \nu} dx^\nu \\
= (dr \, d\theta \, d\phi) \begin{pmatrix} 1 & 0 & 0 \\ r^2 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \\
= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\
= ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\
\text{Contr.} \Rightarrow ds^2 = g_{\mu \nu} \bar{\nu}^\mu = g^{\alpha \mu} (\text{which in this case is} \ g^{\alpha \mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix})
A single space can admit many different metrics (from different coordinate choices).

We know that flat space, e.g., $\mathbb{R}^2$ can have $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ w/ $(x, y, z)$,

but we could have a more complicated $g_{\mu\nu}$ for the same space (think $(r, \theta, \phi)$).

So an important question is "how do we figure out if a space is curved?"

This is hard to answer just by looking at the form of the metric, e.g.,

$\mathbb{R}^2 (r, \theta, \phi) \Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \sin^2 \theta \end{pmatrix}$ Flat!

$S^2 (\theta, \phi) \Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \sin \theta \end{pmatrix}$ Curved!

We will eventually get a good measure for curvature, but to get a hint at what it contains we first recall that any manifold can appear flat in the neighborhood (small region) around a point. This means that by choosing the right coordinates the metric can be brought to the form $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\partial_\alpha g_{\mu\nu} = 0$ near the point in question.

Of $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The coordinates that do this are called Local Inertial Coordinates.
Example of LIC:
Consider $S^2 \equiv \{ (\theta, \phi) \}$

\[
\begin{align*}
&\frac{ds^2}{d\theta^2} + \frac{1}{\sin^2 \theta} \frac{d\phi^2}{\cos^2 \phi} = g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix}
\end{align*}
\]

Let's focus on the north pole, i.e. $\theta = 0 \Rightarrow g_{\mu\nu} \approx \begin{pmatrix} R^2 & 0 \\ 0 & 0 \end{pmatrix}$ which is horribly degenerate.

Let's find better coordinates for this point: $x = R \cos \phi$ near the north pole we want $y = R \sin \phi$ move in $z$, and since $\theta = 0$, we use small angle approx. $\sin \theta \approx \theta$.

Inverting: $\theta = \frac{1}{R} \tan^{-1} \left( \frac{y}{x} \right)$
\[
\begin{align*}
d\theta &= \frac{1}{R} \left( x \frac{dy}{y} + y \frac{dx}{x} \right) \\
d\phi &= \frac{1}{\sqrt{x^2+y^2}} \left( x \frac{dy}{y} - y \frac{dx}{x} \right)
\end{align*}
\]

Using: $\sin \left( \frac{1}{R} \tan^{-1} \left( \frac{y}{x} \right) \right) = \frac{\sqrt{x^2+y^2}}{R} - \frac{1}{6} \left( \frac{x^2+y^2}{R^3} \right)^3 + \ldots$

\[
\begin{align*}
&ds^2 = \left( 1 - \frac{1}{3R^2} + \ldots \right) dx^2 + \left( 1 - \frac{1}{3R^2} + \ldots \right) dy^2 + \left( \frac{xy}{3R^2} + \ldots \right) dx \, dy
\end{align*}
\]

\[
\begin{align*}
g_{\mu\nu} &= \begin{pmatrix} 1 - \frac{1}{3R^2} & \frac{xy}{3R^2} \\ \frac{xy}{3R^2} & 1 - \frac{1}{3R^2} \end{pmatrix} + \text{h.c.}
\end{align*}
\]

Note: $g_{\mu\nu} \left( x = y = 0 \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\partial_x g_{\mu\nu} \bigg|_{x=y=0} = 0$

But: In general the second derivatives $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$, $\frac{\partial^2}{\partial x \partial y}$ will not vanish even when $x = y = 0$. These are the quantities from which we will build a good measure of curvature!
In addition to pointing out what to look for in measuring curvature, LICs have the following immensely important use:

Ask a question $\rightarrow$ answer in LIC’s (usually easy) $\rightarrow$ express answer w/ tensors

$\downarrow$

answer good in any coordinates!!

(similar to inert frame use in SR!)

But this is why we have to be careful, e.g. $\Gamma^i_{jz}$ is not a tensor!