So far we have been working with 1 derivatives.

Covariant Derivative: \[ \nabla_\alpha V = \lim_{\lambda \to 0} \frac{V^\alpha(x^\alpha + \lambda \delta x^\alpha) - V^\alpha(x^\alpha)}{\lambda} \]

Directional Covariant Derivative: \[ \frac{\partial V^\alpha}{\partial \lambda} = \lambda \to 0 \frac{V^\alpha(x^\alpha + \lambda \delta x^\alpha) - V^\alpha(x^\alpha)}{\lambda} = \frac{\partial x^\alpha}{\partial \lambda} \nabla_\alpha V \]

Both use 1-transport so both are covariant. The difference is the path.

In \[ \nabla_\alpha V \] we pick a value of \( \mu \) (a coordinate) and the path is a shift along \( x^\alpha \).

In \[ \frac{\partial V^\alpha}{\partial \lambda} \] we shift along the curve \( x^\mu(\lambda) \) which can be arbitrary, i.e. not along coordinate axes! Of course if we know how the curve changes w/ coordinates \( \frac{dx^\alpha}{d\lambda} \), and we know how the vector changes w/ coordinates \( \nabla_\alpha V \), then we can combine these to determine how the vector changes along the curve \( \frac{dx^\alpha}{d\lambda} \nabla_\alpha V \), i.e. \( \frac{dV^\alpha}{d\lambda} \).
Geodesics

Recall: $E + h$ (Maxwell + Lorentz Force) / Newton $\Rightarrow (EE + Geodesics)$

There are 2 ways to define geodesic paths $x^n(t)$:
1) Curves $x^n(t)$ which extremize the distance between two points.
2) Curves $x^n(t)$ which $\parallel$-transport their own tangent vectors.

In $\mathbb{R}^2$:

We know that straight lines are the shortest paths and we can see that the tangent vectors are $\parallel$-transported.

Consider a non-geodesic path in $\mathbb{R}^2$:

Note: Tangent vectors are not $\parallel$!

To really drive the point home consider $\parallel$-transporting a vector along a circle in 2 different spaces:

In $\mathbb{R}^2$:

Note: What started as a tangent vector is no longer tangent!

But notice:

Not the shortest distance between $A \rightarrow B$!

In $S^1$:

The shortest distance between $A \rightarrow B$!
To formalize the definition of geodesics by II-transport, recall that a curve $\gamma^i(\lambda)$ has components $\frac{d\gamma^i}{d\lambda}$ of its tangent vector.

If $\gamma^i(\lambda)$ is a geodesic (call it $\gamma^i_{geo}(\lambda)$) then these components should be covariantly constant along the curve, i.e. II to each other.

Thus: $\gamma^i(\lambda) = \gamma^i_{geo}(\lambda)$ if

$$\frac{d}{d\lambda} \frac{d\gamma^i}{d\lambda} = 0 = \frac{d\gamma^i}{d\lambda} \nabla_{\frac{d\lambda}{d\lambda}} \frac{d\gamma^i}{d\lambda}$$

$$= \frac{d\gamma^i}{d\lambda} \left( \frac{d\gamma^i}{d\lambda} + \Gamma^k_{ij} \frac{d\gamma^k}{d\lambda} \right)$$

The geodesic equation

$$0 = \frac{d^2\gamma^i}{d\lambda^2} + \Gamma^k_{ij} \frac{d\gamma^k}{d\lambda} \frac{d\gamma^i}{d\lambda}$$

To use this first note that it is 2nd order so we need 2 boundary conditions before solving.

We could give an initial position $\gamma^i(\lambda_0)$ and “velocity” $\frac{d\gamma^i}{d\lambda}|_{\lambda_0}$ and then this generates the geodesic “launched” from there.

Alternatively, and more familiar, we could give an initial and final position and this gives the external path between them.

Since $\Gamma^k_{ij}$ depends on $g_{ij}$, the explicit form will vary for different geodesics.

An intuitive example: $\mathbb{R}^2$ with $\langle x, y, z \rangle \Rightarrow \Gamma = 0 \Rightarrow \frac{d\gamma^i}{d\lambda} = 0$ for geodesics

$\downarrow$

$\gamma^i(\lambda) = \lambda \gamma^i + \gamma^i_0$

$\text{constants set by boundary conditions}$

A straight line! Clearly the shortest path in $\mathbb{R}^2$. 

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To see the extremization more explicitly consider another example.

\[ R^1 \omega/ (r, \theta) \rightarrow \lambda^1 = \lambda^1 + \gamma \lambda^1 \rightarrow \Gamma^r_{\theta \theta} = -r, \Gamma^\theta_{\theta r} = \Gamma^\theta_{r \theta} = \frac{1}{r} \]

Parameterize \( \lambda = s \) distance along the curve, i.e., \( \chi^\lambda(s) = (\gamma(s), \theta(s)) \)

Then the total length is:
\[
\gamma^2 = \int_A^B \sqrt{g_{ij} dx^i dx^j} = \int_A^B \sqrt{\frac{dr^2}{r^2} + \frac{d\theta^2}{\theta^2}} ds
\]

Extremizing this is akin to extremizing an action \( S = \int L(x, u) dt \) in \( CE \).

Normally:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial x} = 0
\]

If we call \( \frac{d \gamma}{ds} = V_r, \frac{d \theta}{ds} = V_\theta \) then:
\[
\frac{d}{ds} \left( \frac{\partial L}{\partial V_r} \right) - \frac{\partial L}{\partial r} = \frac{d}{ds} \left( \frac{\partial L}{\partial V_\theta} \right) - \frac{\partial L}{\partial \theta} + \frac{1}{r} \frac{d}{ds} \frac{\partial L}{\partial \theta} = 0
\]

Compare this to the geodesic equation:
\[
\frac{d^2 \gamma}{ds^2} + \Gamma^\gamma_{\gamma \gamma} \frac{d \gamma}{ds} = 0
\]

and in the above and \( \lambda = s \)
\[
\frac{d^2 V_r}{ds^2} + \Gamma^r_{\theta \theta} \frac{d V_r}{ds} + \Gamma^r_{r \theta} \frac{d \theta}{ds} = 0
\]

\[
\frac{d^2 \theta}{ds^2} + \Gamma^\theta_{\theta \theta} \frac{d \theta}{ds} + \Gamma^\theta_{r \theta} \frac{d \gamma}{ds} = 0
\]
If we work in a Lorentzian signature space, e.g., Minkowski, then $E_{\text{KL}} = \int_0^\beta \sqrt{-ds^2}$ and timelike geodesics actually minimize the spacetime length.

To appreciate this consider a geodesic (constant velocity or at rest) object $U$ in $\mathbb{M}$ and an accelerated non-geodesic object $V$.

For $U$: $S_{\text{KL}} = \int_{t_0}^{t_0} ds = t_f - t_0$

For $V$: $S_{\text{KL}} = \int_0^\beta \sqrt{-ds^2} - \int_0^\beta \int_0^{1-u^2} dt = \int_0^\beta \int_1 - u^2 dt + \int_0^\beta \int_1 - (c^2 - u^2) dt$

$= \int_0^\beta - u^2 dt < t_f - t_0$

**B.T.W.** These two trajectories are representative of those in the twin paradox. The twin who remains on Earth follows $U$ and is older than the one that travels away and then back along $V$. 