Conserved Quantities and Spacetime Symmetries

You should be familiar with the following argument from nonrelativistic mechanics:

Euler-Lagrange equations \( \frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = 0 \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = 0 \) if \( L \) does not depend on \( x^i \)

If \( \pi^i \) momentum conjugate to coordinate \( x^i \), then \( \pi^i \) is Constant

This is a simple version of a more powerful argument by Emmy Noether that also applies to field theories and internal as well as spacetime symmetries. Generally: A continuous symmetry of an action gives rise to a conserved current.

Symmetries and conserved quantities are incredibly useful tools for solving equations of motion (think ODEs).

We won't work with a Lagrangian formulation of GR, but if we restrict to spacetime symmetries (isometries) then we can take a slightly different approach.

We will certainly interchange conserved \( \pi^i \) constant, though a distinction should be made!
Normally we make statements of the form “a symmetry in $x$ means that $\frac{dx}{dt} = 0$” which is highly coordinate dependent. We can do something similar in GR.

We know that Li-momenta are given by $P^\mu = m U^\mu = m \frac{dx^\mu}{dt}$ for massive particles.

The geodesic equation can be written:

$$\frac{d}{dt} \frac{dx^\mu}{dt} \nabla_\nu \frac{dx^\mu}{dt} = 0 = P^\nu \nabla_\nu P^\mu \quad (= m \frac{dP^\mu}{dt})$$

Let’s hit both sides with $\nabla_\mu$ and use $\nabla_\nu g_{\mu \nu} = 0$ (metric compatibility):

$$0 = P^\nu \nabla_\nu P^\mu = P^\nu (\partial_\nu P^\mu - \Gamma^\mu_{\nu\alpha} P^\alpha)$$

$$= P^\nu \partial_\nu P^\mu - P^\nu \Gamma^\mu_{\nu\alpha} P^\alpha$$

$$= m \frac{dP^\mu}{dt} - \frac{1}{2} \left( \partial_\nu g_{\mu \alpha} - \partial_\alpha g_{\mu \nu} - \partial_\mu g_{\nu \alpha} \right) P^\nu P^\alpha$$

$$= m \frac{dP^\mu}{dt} - \frac{1}{2} \left( \partial_\nu g_{\mu \alpha} - \partial_\alpha g_{\mu \nu} \right) P^\nu P^\alpha$$

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Hence we find that $\frac{dP^\mu}{dt} = 0$ if $\partial_\nu g_{\mu \alpha} = 0$ (or if $g_{\mu \nu}$ is independent of $x^\alpha$).

This is a conservation result similar to $\frac{dx^\mu}{dt}$ is conserved if $L$ is independent of $x$. 
But we can do better! So far our discussion has been very coordinate dependent. We can say that for a given choice of coordinates, if the metric is independent of one coordinate then there is a corresponding conserved momentum. But we would like a coordinate independent way of identifying symmetries (and conserved quantities).

Suppose that \( p^6 \) is a conserved momentum component, i.e. \( \frac{dp^6}{dt} = 0 \).

We will introduce a vector \( K^m \) such that \( K^m \, p_m = p^6 = K^n \, p^n \).

Then: \( \frac{dp^6}{dt} = 0 = \frac{d}{dt} (K^m p^m) = \frac{dx^m}{dt} \nabla_m (K^n p^n) = \frac{dx^m}{dt} \nabla_m (K^n p^n) \)

(same when acting on scalars!)

Multiplying by \( m \): \( 0 = p^n \, p^m \nabla_n K_m + K_n \, p^n \nabla_n p^n \)

\[ 0 = p^n \, p^m \nabla_n K_m + K_n \, p^n \nabla_n p^n \]

(by geodesic equation)

\[ 0 = p^n \, p^m \nabla_n K_m + K_n \, p^n \nabla_n p^n \]

(\( \nabla_n K_m = 0 \) since \( S \) is a Killing vector)

Then we find that if \( \nabla_n K_m = 0 \), then \( K_m \) is a symmetry of the geometry and \( K^m \) is a conserved quantity.
3.9

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How many solutions to $\nabla (nKu) = 0$ should we expect?

In general, this is hard to know in advance, but for maximally symmetric spaces we have a simple answer.

Recall that locally any manifold looks like $\mathbb{R}^n$ or $\mathbb{M}^n$. These allow:

- translations
- $\frac{1}{2}n(n-1)$ rotations ($\mathbb{R}^n$)
- Lorentz tran.$\mathbb{M}^n$

Together these form the Euclidean or Poincaré groups.

In total: $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ local symmetries

$
\text{In } 4D
$

If all of the local symmetries are also valid globally, then the space (time) is called maximally symmetric and we should expect $\frac{1}{2}n(n+1)$ independent solutions to $\nabla (nKu) = 0$. 
You might guess that only $\mathbb{R}^n$ or $\mathbb{M}^n$ themselves are maximally symmetric, but that's not quite the case.

For $S^1$ we know (via its embedding into $\mathbb{R}^2$) that it is symmetric under rotations around $x, y, z$. So we have $\mathcal{S} = \frac{\mathbb{Z}}{2} \mathbb{Z}^{k+1}$ symmetries.

To see the local translational breakdown, consider the north pole ($P$). Then: $R_z$, is the single $\frac{1}{2} \mathbb{Z}^{k-1}$ rotation. $R_y, R_x$ are the $2$ translations.

$S^2$ is maximally symmetric, even though it definitely isn't $\mathbb{R}^2$ (it's curved!)
Maximally symmetric spaces do not have to be flat, but their curvature does take a simple form.

Due to the translation invariance, if we know $R^\gamma_\mu\nu\omega$ at any point, it must have the same value at any other point, i.e. $R^\gamma_\mu\nu\omega(x) = R^\gamma_\mu\nu\omega = \text{constant}$.

In fact, just knowing the Ricci scalar and the metric one can show:

For maximally symmetric spaces: $R^\gamma_\mu\nu\omega = \frac{b}{n(n-1)} (\eta_{\gamma\omega} \eta_{\mu\nu} - \eta_{\mu\nu} \eta_{\gamma\omega})$ (no derivatives!)

\[\text{Note: antisym \{ } \eta_{\alpha\beta} \text{\} }\]

\[\epsilon_{\mu\nu\rho} \eta_{\eta_{\beta\gamma}} = \epsilon_{\eta_{\alpha\beta}} \eta_{\mu\nu\rho}\]

A catalog of maximally symmetric spaces:

<table>
<thead>
<tr>
<th>Euclidean</th>
<th>Lorentzian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R=0$</td>
<td>$\text{M}^n$, $\text{P}^n$ (Euclidean, tori)</td>
</tr>
<tr>
<td>$R&gt;0$</td>
<td>$S^n$ (spheres)</td>
</tr>
<tr>
<td>$R&lt;0$</td>
<td>$H^n$ (hyperbolic)</td>
</tr>
</tbody>
</table>
Let’s go through all the gory details for $S^3$.

$L$ : $d^3 = d\theta + \sin \theta \, d\phi \, d\lambda \quad g_{\mu
u} = \delta_{\mu\nu} \quad g_{\phi \phi} (\sin \phi \, \sin \theta)

\Gamma^\phi_{\phi \theta} = \cot \theta \quad \Gamma^\phi_{\theta \phi} = - \sin \phi \cos \theta \quad \text{all other } \Gamma^\phi = 0$

We expect 3 ind. solutions to $\nabla_\mu K^\mu = 0 = J_\mu K^\mu + \partial_\lambda K^\lambda - \Gamma^\lambda_{\mu \nu} K^\nu$

\[
\begin{align*}
\lambda = \nu = \theta & \quad 0 = \partial_\theta K^\theta - \Gamma^\lambda_{\theta \phi} K^\phi = \partial_\theta K^\theta \\
\lambda = \nu = \phi & \quad 0 = \partial_\phi K^\phi - \Gamma^\lambda_{\phi \theta} K^\theta = \partial_\phi K^\phi + \sin \phi \cos \theta K^\theta \\
\lambda = \theta, \nu = \phi & \quad 0 = \partial_\phi K^\phi + \partial_\theta K^\theta - \Gamma^\lambda_{\phi \theta} K^\lambda = \partial_\phi K^\phi + \partial_\theta K^\theta - \cot \theta K^\theta
\end{align*}
\]

3 ind. solutions (each solution satisfies all 3 equations above!) are:

$K^1_\mu = (\cos \phi, \sin \phi, 0)$

$K^2_\mu = (\sin \phi, \frac{1}{2} \cos \phi \sin \theta)$

$K^3_\mu = (\cos \phi, -\frac{1}{2} \sin \phi \sin \theta)$

Each of these corresponds to a conserved quantity. If $P^\mu = (p^\theta, p^\phi)$ then:

\[
\begin{align*}
K^1_\mu P^\mu &= \sin \phi \cos \theta P^\phi \\
K^2_\mu P^\mu &= \sin \phi P^\phi + \frac{1}{2} \cos \phi \sin \theta P^\theta \\
K^3_\mu P^\mu &= \cos \phi P^\phi - \frac{1}{2} \sin \phi \sin \theta P^\theta
\end{align*}
\]

All are conserved, i.e. \( \frac{\partial (K_\mu P^\mu)}{\partial \lambda} = 0 \); $L_\lambda = 0$ trans on $S^3$

$P^\theta$ forms in $S^3$

$P^\phi$ around $\theta = 0$