In our first attempt at finding an exact solution to Einstein’s equations, we will seek spherically symmetric solutions. This is a useful approach because:

a) It is way easier than "less symmetric solutions."
b) It is obviously relevant for astrophysical applications, e.g., stars, planets, etc.

Let’s warm up with a similar exercise from $\mathbb{R}^4$.

Consider Gauss’ law: $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$

Suppose we want a spherically symmetric solution for $\mathbf{E}$ in a region where there is no charge density, i.e., $\rho = 0$, e.g., “outside” of a charged object. $\mathbf{E}$ only depends on $r$.

Then we need to solve $\nabla \cdot \mathbf{E} = 0$ with a spherically symmetric ansatz $\mathbf{E} = f(r) \hat{r}$.

"Vacuum" form of Gauss' law.

Thus: $\nabla \cdot \mathbf{E} = \frac{\partial}{\partial r} \left( r^2 f(r) \right) = 0 \Rightarrow f(r) = \frac{K}{r}$

To interpret this, we use an old trick:

\[ \int \nabla \cdot \mathbf{E} \, dV = \int \mathbf{E} \, dV \]

Integrate both sides over the volume of a sphere.

\[ \int E \, dV = \frac{\text{charge}}{\varepsilon_0} \quad \text{using the divergence theorem} \]

Write $\nabla \cdot \mathbf{E} = \frac{\text{charge}}{\varepsilon_0} \Rightarrow E = \frac{\text{charge}}{4\pi \varepsilon_0 r^2} \Rightarrow E = \frac{\text{charge}}{4\pi \varepsilon_0 R^2} \hat{r}$

This works when $\rho$ is extended or point-like. Of course, if $\rho$ is extended with radius $R$, we have to find an “interior” solution for $r < R$.

Notice that solving this so easily was greatly facilitated by choosing the right coordinates.

Of course, we could have started in any coordinate system we liked, but the process would have been much uglier. Along the way, we might have seen how various coordinate redefinitions could simplify the process and eventually get ourselves to spherical polar. Of course, once we have a solution in any coordinate system, the hard work is done and we can take the solution (not the differential equation) and transform it to any coordinate system we want.
Turning to Einstein's equations, we would like to find a spherically symmetric solution in a source free region, i.e. $T_{\mu\nu} = 0$.

Recall:

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ EE

$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$ EE in trace-reverse form

Obviously the latter form is more useful in this situation since $T_{\mu\nu} = 0 \Rightarrow T = 0$ and it reduces to:

$R_{\mu\nu} = 0$ ("vacuum" form of EE)

Ultimately we want to solve this for $g_{\mu\nu}(x)$, which contains 10 independent functions:

$\phi$, $\psi$, $\phi$, $\psi$, $\phi$, $\psi$, $\phi$, $\psi$, $\phi$, $\psi$

I have already assumed that we will use something like spherical polar coordinates $(t,r,\theta,\phi)$. We will find that this is nonetheless still not the simplest coordinate system in which to find (or even present) a solution. We might expect this since our naive sense of spherical polar is based on flat-space, and we expect our solution to be curved.
Of course spherical symmetry will have a lot to say about the 10 unknown functions.

One way to impose spherical symmetry is by imagining the spacetime in terms of an $S^2$ foliation. This means we build up the spacetime by stacking concentric $2$-spheres (like an onion), except in this case they are stacked along a radial direction $r$ (so moving along $r$ changes the "size" of each $S^2$) and lined up along $t$.

This has some immediate consequences:

1) If we focus on one of the $2$-spheres (fix $r$ and $t$) then the metric should take the form $ds^2 = dr^2 + r^2 d\Delta \theta = g_{\rho \rho} = 0$ and we know that the rest of the metric components should not depend on $\theta$ or $\phi$.

2) If we align our $S^2$ shells so that radial geodesics go through the same value of $\theta$ and $\phi$ on each slice.

Additionally, if we stuck them along $t$ such that motion purely along $t$ keeps the value of $\theta$, $\phi$ unchanged.

So spherical symmetry eliminates 5 of the unknown functions, leaving 5 and tells us that nothing else depends on $\theta$ or $\phi$.

So we have: $ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + C(r,t)d\theta^2 + D(r,t)d\phi^2$

Carroll is a bit more careful about this term but we won't worry about that complication.
So we have 4 unknown functions $A$, $B$, $C$, $D$. But we can say more.

Let's redefine $r \rightarrow r' = \sqrt{B(r', t)} r$, which can be inverted to find $r(r', t)$.

Thus: $ds^2 = -A(r', t) dt^2 + B(r', t) dr'^2 + C(r', t) d\theta^2 + r'^4 d\Omega^2$

This is very similar to choosing a gauge to help simplify solving Maxwell's equations for $\mathbf{A}$ and $\mathbf{E}$.

Assuming this redefinition, we will drop the primes henceforth.
Exploiting this gauge freedom even further consider redefining time as
\[ t \rightarrow t' = t - f(r, t') \] \[ \Rightarrow t = t' + f(r, t') \]

Then:
\[ dt = dt' + \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial t'} dt' \]
\[ = \frac{\partial f}{\partial r} dr + (1 + \frac{\partial f}{\partial t'}) dt' \]
\[ dt' = \left( \frac{\partial f}{\partial r} \right) dr + \left( 1 + \frac{\partial f}{\partial t'} \right) dt' \]

And putting this back into the metric we find:
\[ ds^2 = -A(r, r') \left[ \left( \frac{\partial f}{\partial r} \right)^2 dr^2 + \left( 1 + \frac{\partial f}{\partial t'} \right)^2 dt'^2 \right] + 2B(r, r') \left( \frac{\partial f}{\partial r} \right) dr dt' \]
\[ + C(r, r') dr^2 + r^2 d\Omega^2 \]

Collecting the \( dr dt' \) cross-terms:
\[ B \left[ -A(r, r') \left( \frac{\partial f}{\partial r} \right) dr + \frac{\partial f}{\partial t'} dt' \right] \]
\[ = 0 \text{ for } f(r, t') \text{ such that } \frac{\partial f}{\partial r} = \frac{B(r, t')}{A(r, t')} \Rightarrow f(r, t') = \int \frac{B(r, t')}{A(r, t')} + g(t') \]

Assuming that we redefine \( t \rightarrow t' \) using just such a function \( f(r, t') \) then the metric becomes:
\[ ds^2 = \left[ A(r, r') \left( 1 + \frac{\partial f}{\partial t'} \right)^2 \right] dt'^2 \]
\[ + \left[ -A(r, r') \left( \frac{\partial f}{\partial r} \right)^2 + 2B(r, r') \frac{\partial f}{\partial r} + C(r, r') \right] dr^2 \]
\[ + r^2 d\Omega^2 \]

Dropping the primes this is essentially:
\[ ds^2 = \tilde{g}_{tt} \left( \frac{\partial f}{\partial r} \right) dt^2 + \tilde{g}_{rr} dr^2 + r^2 d\Omega^2 \]
\[ \text{1 unknown function } f \]
But we can say even more about these unknown functions \( g_{tt}, g_{rr} \).

If we assume \( A(r, t), C(r, t) > 0 \) [that the original metric was Lorentzian] then we can conclude:

\[
\begin{align*}
    g_{tt} &= -\frac{A}{A} + \frac{B}{A} + C = \frac{B}{A} + C > 0 \quad \text{w/o any assumption for } B(r, t)
    \\
g_{rr} &= -\frac{A}{A} + \frac{D}{A} + C = \frac{D}{A} + C > 0
\end{align*}
\]

To encode what we know about the signs into the unknown functions we can write:

\[
\begin{align*}
    g_{tt} &= -e^{\varphi(r, t)} \\
    g_{rr} &= e^{\varphi(r, t)}
\end{align*}
\]

Everything so far has only used spherical symmetry and gauge (coordinate) choices. We haven’t even used the Einstein equation \( R_{\mu\nu} = 0 \)!

Taking our metric so far and cranking out \( R_{\mu\nu} \), we find that most terms are already 0, and the only nontrivial ones are \( R_{t} t, R_{rr}, R_{\varphi \varphi}, R_{\theta \theta}, R_{\phi \phi} \). To satisfy \( EE \), we must make sure that these nontrivial terms vanish as well and this will further shape our solution.