Rotating Sources and the Kerr Geometry

Consider starting with a round (spherically symmetric) object and then spinning it, giving it nonzero angular momentum $J$ about an axis through its center.

The original object would induce a Schwarzschild geometry $ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2$

The spinning object will induce a different geometry first described by Kerr.

Two differences to anticipate:

* Spinning (at a constant rate) will still be time-independent (static), but not spherically symmetric (stationary) so we should expect $dt \, dx$ cross-terms.

* If the spin axis is aligned with the poles, then we should expect $\theta$-dependence from the "squashing" of the sphere, i.e. $\sin \theta$.

Notes: we still have $\theta$-dependence

Note: This is not cylindrical (no translation invariance along a "z"-axis)

After some soul searching (1963):

$$ds^2 = -(1 - \frac{2Mr}{\alpha^2})dt^2 + \frac{2Mr}{\alpha^2} dr^2 + \rho^2 d\Omega^2 + \sin^2 \Theta \left[ (r^2 + a^2)^2 - r^2 \sin^2 \Theta \right] d\Theta^2$$

where:

\[ \{t,r,\Theta,\phi\} \] Boyer-Lindquist coordinates

\[ \alpha = \frac{r^2 + a^2 - M\sqrt{r^4 - a^4}}{2Mr} \]

\[ \Theta(r,\phi) = \frac{r}{\sqrt{r^2 + a^2}} \cos \phi \]

To note:

* For $a \rightarrow 0$ $ds^2 \rightarrow ds^2_{\text{Schwarzschild}}$

* For $r \rightarrow \infty$ (with $M, a$ fixed) $ds^2 \rightarrow -dt^2 + dr^2 + r^2 d\Omega^2$ (Asymptotic Flatness)

* For $M \rightarrow 0$ (with $a$ fixed) $ds^2 = -dt^2 + \frac{r^2 + a^2}{r^2 + a^2} dr^2 + (r^2 + a^2)^2 d\Theta^2 + \sin^2 \Theta d\Phi^2$

which is really just $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

where: $x = \frac{r^2 + a^2}{\sqrt{r^2 + a^2}} \sin \Theta \cos \Phi$

$y = \frac{r^2 + a^2}{\sqrt{r^2 + a^2}} \sin \Theta \sin \Phi$

$z = r \cos \Theta$

Oblate spheroidal coordinates

since it is rotating in $\phi$
We notice that the metric is badly behaved when $\beta=0$ and when $\Delta=0$. Let's investigate these in turn.

$\beta=0$ is a true curvature singularity (a curvature scalar blows up), but it is an interesting one.

$\beta^2 r^2 + a^2 \cos^2 \theta = 0$ requires both $r=0$ and $\theta = \frac{\pi}{2}$. Contrast with Schwarzschild where $r=0$ blows up for any $\theta$ or $\phi$.

Before we look at $\beta=0$, let's actually look at $r=0$ (which turns out to be nontrivial).

$$ds^2|_{r=0} = -dt^2 + \frac{(a \cos \delta \theta)^2 + (\sin \delta \theta)^2 d\theta^2}{\delta^2 + \sin^2 \delta \theta^2}$$

with $\delta = a \sin \theta$, $\theta \in [0, \pi]$ and $\delta \in [0, \pi]$.

So $r=0$ is actually a 2+1 D volume which is cylindrical in $\{t, \delta, \phi\}$. Compare to Schwarzschild where $ds^2|_{r=0} = -dt^2$, i.e. a line along $t$.

Okay, that's weird, but getting back to $\beta=0$, we need $\delta = \frac{\pi}{2}$ which is $\theta = \pi$, i.e. the outer edge of the cylinder. So our singularity is spatially a "ring" which is then extended in time.

Note: The region inside of the ring is nonsingular. You could pass through it without dying.

Should this be expected? Well yes, sort of. We know that our source had 2 parameters, $M$ and $J_\phi$. If the singularity was pointlike, we could encode $M$, but what about $J_\phi$?

Now we see that $J_\phi$ is encoded in the deformation of the point to a ring, analogous to the deformation of the sphere to the spinning oblate spheroid. In fact the ring is extended around $\phi$.

You may have guessed that $J_\phi$ could be encoded in the "intrinsic spin" of the singularity, but GIL is classical and doesn't know about that nonsense!
Okay, what about $\Delta = 0$? First of all it is not a curvature singularity. In fact, much like with Schwarzschild, it indicates the presence of a horizon.

But: $\Delta(t) = r^4 - 2Mr + \epsilon^2 = 0 \Rightarrow r^2 = GM \pm \sqrt{GM^2 - \epsilon^2}$ (horizons)

Let's vary $\epsilon$ and see what happens. But first, note that the sign of $\Delta$ determines the sign in front of $dt^2$. For $\Delta > 0$, $r$ is spatial and can increase or decrease, but if $\Delta < 0$, $r$ is spacelike so only moves in one direction (compare to Schwarzschild case).

- $\epsilon = 0$ \( r = 0 \) (the Schwarzschild singularity)
  \( r^4 - 2Mr = 0 \) (the Schwarzschild horizon)

- $\epsilon < GM$ \( \text{"extremal"} \) \( r > r_+ > 0 \)

- $\epsilon = GM$ \( \text{"extremal"} \) \( r = r_+ = GM \)

- $\epsilon > GM$ \( \text{"over-extreme"} \) no horizon $\Rightarrow$ a naked singularity

You saw this in your HW!
To maximaliy extend the Kerr geometry, we could adopt Kruskal-type coordinates, but let's go ahead and set up a more powerful tool.

Conformal (or Penrose) diagrams:

\[ M^{-1} ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad t \in (-\infty, \infty), \quad r \in (0, \infty) \]

Let's get infinity and swag up to it with

\[ T = \tan^{-1}(t + r) + \tan^{-1}(t - r) \quad \text{with finite ranges} \]
\[ R = \tan^{-1}(t + r) - \tan^{-1}(t - r) \quad |T| < \pi - R \]

Then:

\[ ds^2 = \frac{1}{(\cos T + \cos R)^2} \left( -dt^2 + dr^2 + \frac{r^2}{\omega(T, R)} d\Omega^2 \right) \]

or

\[ ds^2 = \frac{1}{\omega(T, R)} d\Omega^2 \]

confomrally related geometry (preserves angles, in particular light-cones!!)

To visualize it, suppose \( \Phi \in \Phi \) (purely radial motion) then:

\[ T^2 = \pi, \quad R = 0 \]
\[ T = 0, \quad R = \pi \]
\[ 2T = \pi, \quad R = 0 \]

Unwrapping:

\[ t = \text{time-like} \quad \text{future} \]
\[ t = \text{time-like} \quad \text{past} \]
\[ r = \text{space-like} \quad \text{future} \]
\[ r = \text{space-like} \quad \text{future} \]
\[ \gamma^+ = \text{future null} \quad \text{All light-like geodesics begin and end here} \]
\[ \gamma^- = \text{past null} \quad \text{All light-like geodesics begin and end here} \]