Consider an object outside of \( r_{\text{ergo}} \) with \( E_0 > 0 \). It enters the ergosphere and splits into

\[ E_0 = E_i + E_k \Rightarrow E_i > E_0 > 0 \]

so \( E_i \) can leave the ergosphere, but the \( E_k \) is trapped.

To an outside observer, an object with \( E_0 \) approaches the BH, delivers negative energy \( E_k \) (or subtracts off energy \( - E_k \)) and then leaves. Now the energy \( E_k \) includes contributions from the rest mass and motion of the object, but once it is absorbed and included in the energy of the BH, as discussed earlier, it must now appear as a contribution to the mass of the BH. In other words \( E_k = \frac{2M}{r_{\text{in}}} \). But since \( E_k < 0 \), we are reducing the mass of the BH.

Now to get this process to work, i.e. to have \( E_k \) be absorbed and \( E_i \) be on a trajectory that leaves the ergosphere, one can show that \( \frac{\mathbf{j}_k}{\mathbf{r}_{\text{in}}} \leq \frac{E_k}{\mathbf{r}_{\text{in}}} \), where \( \mathbf{r}_{\text{in}} = r_{\text{in}}^2 + a^2 \) is the angular velocity of the horizon. Since \( E_k < 0 \) this tells us that \( \mathbf{j}_k \) and \( \mathbf{r}_{\text{in}} \) have opposite signs, hence the absorbed object must have angular momentum opposite in direction of the BH. But this means that absorption of \( \mathbf{j}_k \) reduces the angular momentum of the BH by

\[ 5\mathbf{r}_{\text{in}} = \mathbf{j}_k. \]

One important thing to remember is that this process requires an ergospheric region. Therefore the limit of this is when \( r_{\text{in}} = 0 \) or Kerr = Schwarzschild, or \( r_{\text{in}} \rightarrow \infty \) beyond which nothing escapes.

For a perfect Penrose process \( \mathbf{j}_k = \frac{E_k}{\mathbf{r}_{\text{in}}} \), or \( 5\mathbf{j} = \frac{5M}{\mathbf{r}_{\text{in}}} \) (chapter for BH, henceforth just \( 5\mathbf{j}, 5M \))

So we start with \( M_{\text{Kerr}} \) and reduce it with Penrose process to \( M_{\text{Schwarzschild}} \). Shouldn't the horizon area get smaller? Not so fast.

First of all, since \( r_H = M_{\text{Schw}} + \sqrt{M_{\text{Schw}}^2 - a^2} \) where \( J = Ma \). From this, since both \( M \) and \( J \) are decreasing, it isn't immediately obvious what is happening to \( r_H \). However, we really should be considering the horizon area which is nontrivially related to \( r_H \) by the geometry.

\[ A_H = \frac{1}{2} \int_{r_H}^{\infty} Y_{ij} - \text{induced metric on horizon from } ds^2 \text{ with } r = r_H, \quad dr = dt = 0 \]

\[ \approx \frac{4}{\pi} \left( r_H^2 \frac{a^2}{a^2} \right) \]

\[ \approx \frac{8\pi \sqrt{G^2 M^2 - G^2 M^2 a^2}}{J^2} \]
If we vary $\delta M$ and $\delta J$ we find:

$$\delta A_\mu = \frac{8 \pi a}{\rho \sqrt{a^2 - \rho^2}} \left( \delta M - \frac{\delta J}{\rho} \right)$$

Recall that $\delta J \leq \frac{\rho}{\rho}$

So $\delta A_\mu > 0$ and the area theorem is (nontrivially!) obeyed.
The absolute nondecreasing of $A_H$ sounds a bit like entropy. In some sense, we should expect a BH to have some observable entropy (we can observe $A_H$) since:

**Bekenstein:** If a BH had no observable entropy, we could take an external system with zero entropy $S_{\text{ext}}$ and upon throwing it into the BH, decrease the entropy of the observable universe, thus violating the 2nd Law of Thermodynamics.

To preserve the 2nd Law, BHs must admit an observable entropy. One might think to use the mass $M$ of the BH to calculate $S$, but the Penrose process allows $M$ to decrease in certain cases. It is only $A_H$ which is a suitable proxy for entropy.

But this already tells us something deep. While entropy usually scales with the volume of a system, in this case it scales with the area. This points to the holographic nature of gravity, since information from 4D is captured in a 2D surface.
Quantifying this idea, from the Kerr case:

$$\delta S = \frac{K}{8 \pi G} \delta A + \oint_{\partial \mathcal{H}} \frac{\delta J}{2 \partial \mathcal{H}} \delta J$$

w/ $K = \frac{\Delta \mathcal{H} - \alpha}{2 \pi G (\Delta \mathcal{H} + \Delta \mathcal{H}^* - \alpha)}$.

$K$ is the “surface gravity” of the BH, or roughly how strong the gravitational pull is near the horizon.

Comparing to $dE = TdS - \partial dU$ we would associate:

$$E = H$$

$$- \partial dU = \oint_{\partial \mathcal{H}} \frac{\delta J}{2 \partial \mathcal{H}}$$

$$\frac{T}{2} dS = \frac{K}{8 \pi G} \delta A$$

It is tempting to identify $T = \frac{K}{8 \pi G}$ and $dS = \delta A$, but in truth the split isn’t obvious.

Enter Hawking: Hawking considered QFT in the curved geometry near the horizon of a BH. Note: He was not doing quantum gravity (which is still not completely understood), he was doing perfectly well-defined QFT with minimal coupling.

Even if you don’t trust QFT in curved space, you could make do w/ QFT in flat space and then apply the equivalence principle.

The Unruh effect is the consequence of uniform acceleration in flat space. A uniformly accelerated observer in flat space experiences not $1^\text{st}$, but rather a Rindler spacetime. Quantizing a field in terms of Rindler time is very different than w/ Minkowski time (we use time to identify allowable positive frequency modes from $\omega^4$ dependence). The end result is that a uniformly accelerated observer in the vacuum (no particles) of $1^\text{st}$ actually sees a thermal distribution of all allowed particle types (dominated by lowest mass) coming at them w/ the temperature $T = \frac{\kappa}{2\pi}$ where $\kappa$ is the acceleration of the observer.

But now we use the equivalence principle to replace $a = K$ and find $T = \frac{\kappa}{2\pi}$

and hence $dS = \frac{\delta A}{4\pi \kappa} \Rightarrow S = \frac{A}{4\pi \kappa}$.
Black Hole Evaporation

Perhaps the most surprising aspect of Hawking's result is that BHs seen to radiate. The cartoon version of this is:

\[ \text{Quantum Fluctuation as particle/antiparticle pair.} \]

Why surprising? Conservation of energy implies that the BH is losing energy and hence mass in this process and (as easily seen for Schwarzschild) this means the horizon area is decreasing. Does this violate the area theorem? Actually no! The area theorem assumed a weak energy condition ($\rho \geq 0$), but these quantum fluctuations can have $\rho < 0$, hence violating the w.e.c.

So BHs can evaporate! Of course for large BHs this rate is small especially compared to any accretion. However for microscopic BHs it leads to them being very short-lived.

Black Hole Information Paradox

Perhaps the most perplexing thing about BH evaporation is the seeming loss of information. First note that in principle, in an otherwise empty universe and given a long enough time, any BH will completely evaporate.

Now consider 2 non-rotating, equal mass cosmic:  

\[ \text{At this stage you could} \]

\[ \text{“hidden” behind horizon.} \]

\[ \text{Uh-oh... no more horizon.} \]

\[ \text{only } T_c \text{ radiation which} \]

\[ \text{is identical!} \]

Resolving this puzzle is almost certainly going to require a well-understood quantum theory of gravity!