$$U^a = \frac{dx^a}{d\xi} \text{ \textit{weirdness:}} \quad \eta_{ab} U^a U^b = -1$$

$$\eta_{ab} = \frac{dx^a}{d\xi} \frac{dx^b}{d\xi}$$

$$= \frac{dx^a}{d\xi} \frac{dx^b}{d\xi}$$

$$= \frac{dx^a dx^b}{d\xi^2}$$

$$= \frac{dx^a dx^b}{d\xi^2}$$

$$= -1 \quad \text{WTF?}$$

$$\eta \equiv \gamma^2$$

This might seem strange, usually $V_i V^i = V^2$ for $V \equiv c t$.

But consider: $V \cdot V = \left( \frac{dx}{d\xi}, \frac{dy}{d\xi}, \frac{dz}{d\xi} \right)$

$\frac{d\xi}{d\xi} = \frac{dx}{d\xi} + \frac{dy}{d\xi} + \frac{dz}{d\xi} = \frac{d\xi}{d\xi} = 1$

So this is just a consequence of how we are parameterizing paths.

Components of $U^a$:

$$U^0 (z) = \frac{dx^0}{d\xi} = \frac{dz}{d\xi} = \gamma \frac{d\xi}{d\xi} = \gamma$$

$\text{in } \mathcal{S} \text{ w/ } (x_1, y_1, z_1)$

$$U^1 (z) = \frac{dx^1}{d\xi} = \frac{dx}{d\xi} = \frac{d\xi}{d\xi} = V\gamma$$

$\mathcal{S} \rightarrow \mathcal{S}_{rest}$

$$U^a = \begin{pmatrix} \gamma \\ \gamma \gamma \\ 0 \end{pmatrix}$$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\gamma \rightarrow U \cdot U \gamma = -\gamma^2 + \gamma^2 v^2 = -1 + \frac{v^2}{c^2} \rightarrow U \cdot U = -1$

In $\mathcal{S}_{rest}$:

$$U^a = (0) \quad \text{(again: } U \cdot U = -1)$$
For momentum: 
\[ p' \rightarrow p' = m\gamma \frac{E}{\gamma} \]

If \( v < c \) then \( \gamma = 1 + \frac{v^2}{2} + \cdots \)

\[ \begin{aligned}
\gamma \approx 1 + \frac{v^2}{2c^2} \quad \text{rest energy}
\end{aligned} \]

\[ \begin{aligned}
E = m\gamma v = m\gamma v + \rho(v^2) + \cdots \equiv p' \quad \text{non-relativistic momentum}
\end{aligned} \]

So: \( p^\mu = m\gamma v^\mu \) (E)

Then: \( p'_\mu p^\mu = m^2 \gamma v^2 = -m^2 = -E^2 + p^2 \)

\[ E^2 = p^2 + m^2 \quad (\text{with} \quad E = p^0 + m \frac{c^2}{m^2} v^2) \]

Even though our parameterization doesn't work when \( n^0 = 0 \), this result does.

\[ p^\mu = \begin{cases} 
< 0 & n^0 > 0 \quad ds^0 < 0 \quad \text{timelike} \\
= 0 & n^0 = 0 \quad ds^0 = 0 \quad \text{lightlike} \\
> 0 & n^0 < 0 \quad ds^0 > 0 \quad \text{spacelike (tachyonic)}
\end{cases} \]

We could try to relativize \( F^\mu_\nu \) to get \( \Sigma F^\mu_\nu = \frac{dp^\mu}{dt} \) which will combine the

work-energy and impulse-momentum theorems. But our primary concern is

the gravitational force which will play out a bit differently.
The next topic is going to seem like a strange focus at first, but remember that interesting sources of gravity are usually large, so instead of thinking of a particle at a time, we should consider a large number of them. The source of curvature/gravitation will be energy (including mass) and momentum. We already have a quantity describing this, i.e. \( P^\alpha \), but this is really only useful for one particle. We will now see how considering many particles leads us instead to the energy-momentum tensor \( T_{\mu\nu} \).

Densities

\[
\frac{dN}{dV} = \text{scalar density when volume in question is at rest.}
\]

\[
\frac{dN}{dV_{\text{rest}}} = \text{number density when volume contracts or expands.}
\]

We can now ask if \( dN \) as defined is a tensor.

Consider boosting along \( x \) by \( \gamma \): \( dx \rightarrow dx' = \frac{dx}{1 \pm \gamma dx/dc} = \gamma dx = \gamma dx' < dx \)

\[
dN = \gamma dN' \quad \text{if } dN = dx_{\text{rest}}
\]

\( dN \) is clearly not a scalar (it changed!) but neither is a vector, etc. It is not a tensor.

What is it? Well \( dN' = \gamma dN \) is similar to \( dt' = \gamma dt \) but \( dt \) is the \( 0\)-component of a 4-vector. So perhaps \( dN \) is the \( 0\)-component of a 4-vector as well.

\[
dN^\alpha \equiv dN_{\text{rest}} = \left( \begin{array}{c}
\frac{dN}{dx} \\
\frac{dN}{dy} \\
\frac{dN}{dz} \\
\frac{dN}{dt}
\end{array} \right)
\]

\[
\frac{dN}{dV_{\text{rest}}} = \text{constant}
\]

The moral to note: We started \( dN \) as a scalar, but to make a tensor density, we needed to introduce the vector number density \( dN^\alpha \).
Is there a deeper explanation for why creating a tensor density seems to “raise” the tensor nature of what we start with, i.e. $\frac{dN}{\mu} \rightarrow d\mathbf{N}^\mu$ - vector.

For densities we need 3D volumes, but a 3D volume in 4D is labelled by its size $dV$ and direction (think about 2D areas in 3D) which is indicated by a normal dual vector $\mathbf{n} \cdot dV$

$\mathbf{2}$ normal to 3D “surface”

The quick and dirty: In 3D a surface is defined by $f(x,y,z) = 0$;

e.g. $x^2 + y^2 + z^2 = R^2 \Rightarrow f(x,y,z) = x^2 + y^2 + z^2 - R^2 = 0$

Then $dA = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 2x dx + 2y dy + 2z dz = \mathbf{r} \cdot d\mathbf{r}$

defined w/ $\frac{\partial f}{\partial x} \neq 0$; so dual vector components

You are familiar w/ $dN = \frac{dN}{\mu} dV$, which now becomes $dN = d\mathbf{N}^\mu \cdot \mathbf{n} dV$ - scalar.

If we try to extend this to $\frac{dp^\mu}{dV}$ we find $\frac{dp^\mu}{dV}$ = ? $\mathbf{n} \cdot d\mathbf{v} \Rightarrow ? = T^\mu \text{momentum-density}$