1. We found in an example in class that a rotation in the $xy$-plane in $\mathbb{R}^3$ has three distinct eigenvalues and three distinct eigenvectors. Prove that the same is true for any rotation in $\mathbb{R}^3$.

First of all realize that any rotation can be realized by a rotation by an angle around some axis in $\mathbb{R}^3$. That is, even a product of two rotations about different axis gives rise to a rotation around a single axis, e.g. $R_{(0,0,1),90^\circ} R_{(1,0,0),90^\circ} = R_{(1,1,1),120^\circ}$

But this means that any rotation can be obtained from a different rotation by simply rotating the coordinate basis appropriately. But then the original rotation operator $R$ is transformed by the basis rotation $R'$ into a new rotation $R''$ according to $R'' = R'^{-1} RR'$.

But this relation means that the new rotation $R''$ is similar to the original rotation $R$. But similar linear transformations share both the same eigenvalues and the same number of distinct eigenvectors.

Applying this to the situation at hand, since the rotation had three distinct eigenvalues and three distinct eigenvectors, then the rotated version of it has the same three eigenvalues, and also has three distinct eigenvectors (though not the same as the original).

2. Is the following matrix diagonalizable? $M = \begin{pmatrix} 3 & 0 & 0 & -1 \\ -\frac{3}{\sqrt{2}} & 2 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$ Explain your reasoning.

Consider the eigenvalues:

$$\det(M - \lambda I) = (3 - \lambda)(2 - \lambda)(4 - \lambda)(3 - \lambda) + 1(-2 + \lambda)(4 - \lambda)$$

$$= (2 - \lambda)(4 - \lambda)[(3 - \lambda)(3 - \lambda) - 1]$$

$$= (2 - \lambda)(4 - \lambda)[9 - 6\lambda + \lambda^2 - 1]$$

$$= (2 - \lambda)(4 - \lambda)[8 - 6\lambda + \lambda^2]$$

$$= (2 - \lambda)(4 - \lambda)(4 - \lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = 2, \ \lambda_3 = \lambda_4 = 4.$$ 

Now the eigenvectors:

$$Mx_{1,2} = 2x_{1,2} \Rightarrow \begin{pmatrix} 3 & 0 & 0 & -1 \\ -\frac{3}{\sqrt{2}} & 2 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \\ 2d \end{pmatrix}$$

$$\Rightarrow -\frac{3}{\sqrt{2}}a + 2b - \frac{3}{\sqrt{2}}d = 2b \Rightarrow a = d$$

$$4c = 2c \Rightarrow c = 0$$

$$-a + 3d = 2d \Rightarrow a = d$$

$$a = d \Rightarrow b = anything \Rightarrow c = 0 \Rightarrow d = 0$$
\[ M_{x,4} = 2x_{3,4} \Rightarrow \begin{pmatrix} 3 & 0 & 0 & -1 \\ -3/\sqrt{2} & 2 & 0 & -3/\sqrt{2} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \\ 4c \\ 4d \end{pmatrix} \]

\[ \Rightarrow -\frac{3}{\sqrt{2}}a + 2b - \frac{3}{\sqrt{2}}d = 4b \]

\[ 4c = 4c \]

\[ -a + 3d = 4d \]

\[ a = -d \quad b = 0 \quad c = anything \quad d = -a \]

The fact that the geometric multiplicity of \( \lambda = 2 \) is only 1, gives us three total eigenvectors which do not span the space. Therefore the linear operator \( M \) is not diagonalizable.

In fact if you try, one of the closest you will get is \( M' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \).

3. Consider the vector space of even polynomials up to 4th degree armed with the inner product defined by \( (x, y) = \int_0^1 x(t)y(t) \, dt \). A natural basis would be \( X = \{x_0, x_2, x_4\} = \{1, t^2, t^4\} \).

a) Confirm that this basis is not orthogonal.

\[ (1, t^2) = \int_0^1 t^2 \, dt = \frac{1}{3} \neq 0 \quad (1, t^4) = \int_0^1 t^4 \, dt = \frac{1}{5} \neq 0 \quad (t^2, t^4) = \int_0^1 t^6 \, dt = \frac{1}{7} \neq 0 \]

b) Starting with the basis vector \( x_0 \), use the Graham-Schmidt process to determine an orthonormal basis.

\[ y_1 = \frac{x_0}{\|x_0\|} = 1 \]

\[ y_2 = \frac{x_2-(y_1,x_2)y_1}{\|x_2-(y_1,x_2)y_1\|} = \frac{t^2-\int_0^1 t^2 \, dt}{\|t^2-\int_0^1 t^2 \, dt\|} = \frac{t^2-1/3}{\sqrt{\int_0^1 (t^2-1/3)^2 \, dt}} = \frac{\sqrt{45}}{2} (t^2 - \frac{1}{3}) \]

\[ y_3 = \frac{x_4-(y_1,x_4)y_1-(y_2,x_4)y_2}{\|x_4-(y_1,x_4)y_1-(y_2,x_4)y_2\|} = \frac{t^4-\int_0^1 t^4 \, dt}{\|t^4-\int_0^1 t^4 \, dt\|} = \frac{t^4-\frac{5}{6} t^2 + \frac{3}{35}}{\sqrt{64/11025 (t^4 - \frac{6}{7} t^2 + \frac{3}{35})}} = \frac{8}{105} (t^4 - \frac{6}{7} t^2 + \frac{3}{35}) \]

Starting this time with the basis vector \( x_2 \), use Graham-Schmidt to determine an orthonormal basis.

\[ y_1 = \frac{x_2}{\|x_2\|} = \frac{t^2}{\sqrt{\int_0^1 t^4 \, dt}} = \sqrt{5} t^2 \]

\[ y_2 = \frac{x_4-(y_1,x_4)y_1-(y_2,x_4)y_2}{\|x_4-(y_1,x_4)y_1-(y_2,x_4)y_2\|} = \frac{t^4-\sqrt{5} t^2 \int_0^1 \sqrt{5} t^6 \, dt}{\|t^4-\sqrt{5} t^2 \int_0^1 \sqrt{5} t^6 \, dt\|} = \frac{t^4-\frac{5}{2} t^2 + \frac{1}{6}}{\sqrt{\int_0^1 (t^4 - \frac{5}{2} t^2)^2 \, dt}} = \frac{2}{21} (t^4 - \frac{5}{2} t^2) \]

\[ y_3 = \frac{x_0-(y_1,x_0)y_1-(y_2,x_0)y_2}{\|x_0-(y_1,x_0)y_1-(y_2,x_0)y_2\|} = \frac{1-\frac{5}{3} t^2 + \frac{16}{46305} (t^4 - \frac{5}{2} t^2)}{\sqrt{\int_0^1 (1-\frac{5}{3} t^2 + \frac{16}{46305} (t^4 - \frac{5}{2} t^2))^2 \, dt}} = N \left[ 1 - \frac{5}{3} t^2 + \frac{16}{46305} (t^4 - \frac{5}{2} t^2) \right] \]
4. Consider the derivative operators $D = \frac{d}{dt}$ and $D^2 = \frac{d^2}{dt^2}$ acting on the vector space considered in question (3).

(a) Are both of these linear operators on this space? Explain.

The first derivative takes the basis from $\{1, t^2, t^4\}$ to $\{0, 2t, 4t^3\}$. The last two are not elements of the original vector space, so $D$ is not a linear operator.

The second derivative takes the basis from $\{1, t^2, t^4\}$ to $\{0, 2, 12t^2\}$. All of these are elements of the original vector space, so $D^2$ is a linear operator.

(b) For any of these two that are linear operators, find their matrix representations with respect to the three bases discussed in problem (3). Verify the matrix forms by acting on the vector $x = \alpha_0 + \alpha_2 t^2 + \alpha_4 t^4$ (rewritten in terms of each basis) with both the straight derivative(s) and via matrix multiplication of the components.

First of all, $D^2 x = 2\alpha_2 + 12\alpha_4 t^2$ for the first basis so $D^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{pmatrix}$.

For the second basis we have $x = \left(\frac{\alpha_0 + \alpha_2}{3} + \frac{\alpha_4}{5}\right)y_1 + \left(\frac{2}{45}\alpha_2 - \frac{12}{7\sqrt{45}}\alpha_4\right)y_2 + \frac{105}{8}\alpha_4 y_3$ and then $D^2 x = (2\alpha_2 + 4\alpha_4)y_1 + \frac{24}{\sqrt{45}}\alpha_4 y_2$. Then we need a matrix $D^2$ s.t.

$D^2 \begin{pmatrix} \alpha_0 + \frac{\alpha_2}{3} + \frac{\alpha_4}{5} \\ \frac{2}{\sqrt{45}}\alpha_2 - \frac{12}{7\sqrt{45}}\alpha_4 \\ \frac{105}{8}\alpha_4 \end{pmatrix} = \begin{pmatrix} 2\alpha_2 + 4\alpha_4 \\ \frac{24}{\sqrt{45}}\alpha_4 \\ 0 \end{pmatrix}$ \Rightarrow $D^2 = \begin{pmatrix} 0 & \sqrt{45} & \frac{320}{735} \\ 0 & 0 & \frac{192}{105\sqrt{45}} \\ 0 & 0 & 0 \end{pmatrix}$

c) Find the transformation that carries the operators between each of the three basis sets.

To find the transformation we can start by transforming the vector components of $x$ in the first basis to the components in the second basis:

$U \begin{pmatrix} \alpha_0 \\ \alpha_2 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 + \frac{\alpha_2}{3} + \frac{\alpha_4}{5} \\ \frac{2}{\sqrt{45}}\alpha_2 - \frac{12}{7\sqrt{45}}\alpha_4 \\ \frac{105}{8}\alpha_4 \end{pmatrix}$ \Rightarrow $U = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ 0 & \frac{2}{3\sqrt{5}} - \frac{4}{7\sqrt{5}} \\ 0 & 0 & \frac{105}{8} \end{pmatrix}$

Then we find its inverse:

$U^{-1} = \begin{pmatrix} 1 & -\frac{\sqrt{5}}{2} & -\frac{136}{3675} \\ 0 & \frac{3\sqrt{5}}{2} + \frac{48}{735} \\ 0 & 0 & \frac{8}{105} \end{pmatrix}$

And now we transform:

$D^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow UD^2U^{-1} = \begin{pmatrix} 0 & \frac{\sqrt{45}}{3675} & \frac{320}{735} \\ 0 & 0 & \frac{192}{105\sqrt{45}} \\ 0 & 0 & 0 \end{pmatrix}$

5. Find two 2x2 matrices that are both self-adjoint, but which do not commute with each other. Verify that their product is not self-adjoint.
Consider $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $AB = \begin{pmatrix} 1 + i & 1 + i \\ 1 - i & 1 - i \end{pmatrix}$ while $BA = \begin{pmatrix} 1 - i & 1 + i \\ 1 - i & 1 - i \end{pmatrix}$.

The product $AB = \begin{pmatrix} 1 + i & 1 + i \\ 1 - i & 1 - i \end{pmatrix} \neq (AB)^\dagger = \begin{pmatrix} 1 - i & 1 + i \\ 1 - i & 1 + i \end{pmatrix}$. 