1. Consider the matrix \( M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \) acting in \( \mathbb{R}^3 \) with basis \( \{ \hat{i} = (1,0,0), \hat{j} = (0,1,0), \hat{k} = (0,0,1) \} \).

a) (8pts) Is this diagonalizable by a similarity transformation? If so find its diagonal form, it not explain how you determined so.

Let's count the eigenvectors.
Start by finding eigenvalues:
\[
\det[M - \lambda I] = (2 - \lambda)^3 - 1(2 - \lambda) = 0
\]
\[
= (2 - \lambda)(4 - 4\lambda + \lambda^2 - 1) = (2 - \lambda)(\lambda^2 - 4\lambda + 3)
\]
\[
= (2 - \lambda)(\lambda - 3)(\lambda - 1) \Rightarrow \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1
\]
Since each eigenvalue is always guaranteed an eigenvector, then we have three eigenvectors which span the space, and so this is diagonalizable.

The diagonal form would just have the eigenvalues along it: \( M' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

b) (2pts) What would the eigenvalues be for a version of this operator that is similarity transformed by a rotation around the \( \hat{i} \) basis vector by 39° followed by a rotation about the \( \hat{k} \) basis vector by 666°?
Similarity transformations do not change the eigenvalues, so even after the transformation they would still be \( \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1 \).

Turn over for second problem!!
2. Consider the basis \( \{x_0, x_1, x_3\} = \{1, t, t^3\} \) in the space of polynomials consisting of a constant plus linear and cubic terms.

a) (7pts) Going in the order 0 \( \rightarrow \) 1 \( \rightarrow \) 3, Graham-Schmidt the shit out of it. No need to evaluate the normalization factor on the third vector.

\[
y_1 = \frac{x_0}{\|x_0\|} = 1
\]

\[
y_2 = \frac{x_1 - (y_1, x_1)y_1}{\|x_1 - (y_1, x_1)y_1\|} = \frac{t - \int_0^1 t \, dt}{\sqrt{\int_0^1 (t - \frac{1}{2})^2 \, dt}} = \sqrt{2}(t - \frac{1}{2})
\]

\[
y_3 = \frac{x_3 - (y_1, x_3)y_1 - (y_2, x_3)y_2}{\|x_3 - (y_1, x_3)y_1 - (y_2, x_3)y_2\|} = \frac{t^3 - \int_0^1 t^3 \, dt - \sqrt{2}(t - \frac{1}{2}) \int_0^1 \sqrt{2}(t - \frac{1}{2})^3 \, dt}{\sqrt{t^3 - \frac{1}{4} - \frac{9}{16}(t - \frac{1}{2})}}
\]

b) (3pts) Is any derivative operator a linear transformation on this space? If yes, which one(s).

\[
D^2 = \frac{d^2}{dt^2} \text{ is, as well as any derivative of higher order.}
\]