Aiming to generalize our analysis of finite vector spaces, will lead us to the idea of Hilbert space, and all the special functions that live there on.

Getting Started we first need two things:

A closed interval \([a, b]\) is the set of all points \(a \leq x \leq b\). Open \((a, b)\) would mean \(a < x < b\).

A function is square integrable (s.i.) on \([a, b]\) if  
\[
\int_a^b |f(x)|^2 \, dx \text{ exists and is finite.}
\]

Now:

We will begin with function space, that is the space of complex-valued functions of a real variable \(x\), defined on a closed interval \([a, b]\) which are square integrable. It is a vector space.

Recall:

A vector space over a field \(F\) is the set of vectors \(V\) satisfying:

1. \(\{0\}\) forms an abelian group \(\{e\} = 0\)
2. For every \(\alpha \in F\) and \(x \in V\) there exists an element \(\alpha x \in V\) and
   a. \(\alpha (\delta x) = (\alpha \delta) x\) \(\forall \alpha, \delta \in F, x \in V\)
   b. \(1(x) = x\) for all \(x \in V\)
   c. \(\alpha (x + y) = \alpha x + \alpha y\) \(\forall \alpha \in F, x, y \in V\)

In function space, the addition of two vectors is defined as: \((f_1 + f_2)(x) \equiv f_1(x) + f_2(x)\)

and complex scalar multiplication of a vector is defined by: \((\alpha f)(x) \equiv \alpha f(x)\) Not \((\alpha f)(x) = f(\alpha x)\)!!

There are two basic operations that define a vector space. The only worry is closure, i.e. that the sum of two vectors on the space is another vector on the space. In this case meaning is s.i. + s.i. = s.i.?

Well consider:  
\[
|f_1 + f_2|^2 = \left| f_1 + f_2 \right|^2 = \left| f_1 \right|^2 + 2 Re(f_1 \cdot f_2) + \left| f_2 \right|^2
\]

if \(f_1 \cdot f_2 = c\) \(\forall c \in \mathbb{C}\):

\[
\sqrt{(Re(f_1 f_2))^2 + (Im(f_1 f_2))^2} = \sqrt{a^2 + b^2} = \sqrt{a^2 + b^2}
\]

\[
\sqrt{(a^2 + b^2)(x^2 + y^2)} = \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}
\]

so

\[
|f_1 + f_2|^2 = |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2 = |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2
\]

\[
\int_a^b |f_2|^2 \, dx = \int_a^b |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2 \, dx = \int_a^b |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2 \, dx
\]

\[
\int_a^b |f_2|^2 \, dx \leq \int_a^b |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2 \, dx
\]

\[
\int_a^b |f_2|^2 \, dx \leq \int_a^b |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2 \, dx
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\[
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\]

\[
\int_a^b |f_2|^2 \, dx \leq \int_a^b |f_1|^2 + 2 f_1 f_2^* f_2 + |f_2|^2 \, dx
\]
Now part of the above includes products of functions, e.g., \( f_1^* f_2 \). These should be understood as \( f_1^*(x) f_2(x) = \overline{q(x)} \) instead of \( f_1^*(x) f_2(x) \).

Now let's go ahead and define an inner product:

An inner product in a real or complex vector space is a scalar valued function of the ordered pair of vectors \( x \) and \( y \) s.t.

1. \( \langle x, y \rangle = \langle y, x \rangle^* \) [If they are real then order doesn't matter]
2. \( \langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle \) w/ \( a, b \) scalars
3. \( \langle x, x \rangle \geq 0 \) for any \( x \); \( \langle x, x \rangle = 0 \Rightarrow x = 0 \) "positive-definiteness, which can be relaxed"

We can use:

\[
\langle f_1, f_2 \rangle = \int_a^b f_1^*(x) f_2(x) \, dx
\]

Note that i.e., basically uses this for a function \( w \) itself: \( (f, f) = \int_a^b |f|^2 \, dx < \infty \)

We call \( ||f|| = \sqrt{(f, f)} \) the "norm" of \( f \).

Now the inner product of two different functions always exists and is finite:

\[
|f_1^* f_2| = ||f_1|| ||f_2|| \leq \frac{1}{2} (|f_1|^2 + |f_2|^2) \Rightarrow \int_a^b |f_1^* f_2| \, dx \leq \frac{1}{2} (||f_1||^2 + ||f_2||^2) < \infty
\]

The inner product defined above clearly follows all the rules in defining an inner product, except for positive-definiteness.

Clearly \( (f, f) \geq 0 \), but does \( (f, f) = 0 \Rightarrow f = 0 \)?

More precisely, does \( (f, f) = 0 \Rightarrow f(x) = 0 \) for all \( x \)?
Consider the function \( D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \) over the interval \( x \in [0, 1] \).

What is \( \int_0^1 f(x) \, dx \) equal to? First and foremost, it seems that to satisfy the condition above, we need \( \int_0^1 f(x) \, dx \neq 0 \) since \( f(x) \neq 0 \) for some \( x \).

Now \( \int_0^1 f(x) \, dx = \int_0^1 0 \, dx \) since \( f(x) \) is real.

But how do we integrate it? Let's try a good old set of Riemann rectangles.

Recall:

\[
\int_0^1 f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i
\]

Normally, it doesn't matter which we take, max or min,

and in fact, the integral is well-defined if taking either yields the same result.

Now for the function \( D(x) \): Let's suppose we use the rectangle method.

In this case:

\[
\int_0^1 D(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{1}{n} \cdot \max f(x_i) \right)
\]

\[
\int_0^1 D(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{1}{n} \cdot \min f(x_i) \right)
\]

Clearly, for \( \min f(x_i) = \int_0^1 D(x) \, dx = 0 \)

For \( \max f(x_i) = \int_0^1 D(x) \, dx = 1 \)

This means the Riemann integral for the Dirichlet function does not exist!
Can we save it? Yes we can, by redefining the integral into one which can handle all the functions that Riemann can, and more. This is the Lebesgue integral.

Instead of rectangles over dx, Lebesgue instead breaks the area under a curve into horizontal strips.

To get the area of each strip, we start with their heights which we will parameterize generically by t, so the height is dt. For the width, we use the following: \( m(x, f(x) > t) \) which gives us the lengths along \( x \) for regions where the curve is higher than \( t \).

Now if you think about it, \( \int_0^\infty m(x, f(x) > t) \, dt = \int_a^b f(x) \, dx \) where \( m_{ab} \) only does its job over the range of \( x \) from \( a \) to \( b \).

Now we define a measure of integration, and so we can also write \( \int_a^b f(x) \, dx = \int_0^\infty \, dt \).

Here's an important observation: Anytime \( \int_0^b f(x) \, dx \) exists, so does \( \int_a^b f(x) \, dx \), that is wherever the Riemann exists, so does Lebesgue, and then give the same answer.

Even more important: Lebesgue exists even when Riemann's don't. As in the case of Dirichlet:

\[
\begin{align*}
0 & \quad 1 \\
- \infty & \to \infty
\end{align*}
\]

That is, the integral = 0 since the measure of integration dx always = 0. This is called 0 “almost everywhere” and is handled as the 0 function.

Yet another: Lebesgue integrals nicely commute with limits, i.e. \( \lim_{n \to \infty} \int f_n \, dx = \int \lim_{n \to \infty} f_n \, dx \)

\( \int 0 = 0 \Rightarrow f = 0 \) almost everywhere, and our inner-product is well-defined. For Dirichlet, \( \langle f, f \rangle = 0 \Rightarrow D = 0 \) almost everywhere.
We will not need the Lebesgue integral in practice, as the functions we encounter will be Riemann integrable. But we use Lebesgue to define the function space.

Now we need the space to be complete. What does that mean? We should start by saying that completeness here is a statement about the function space, and not so much its vector realization (though we will come to that).

Completeness of a space is the following:
A complete space is one in which there exists no Cauchy sequence of elements of the space which sends towards limits outside of the space.

A Cauchy sequence \( \{x_n\} \) is such that given a number \( \varepsilon > 0 \), then there is some index \( N(\varepsilon) \)
so that if \( m, n > N(\varepsilon) \), then \( ||x_n - x_m|| < \varepsilon \).

An example of an incomplete space. Consider \( X \) to be the space of continuous functions on \([0, 1]\) we norm defined by
\[
||x|| = \int_0^1 |x(t)| \, dt
\]
(note this is different than our space).

Define a sequence of elements of \( X \):
\[
x_n(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\
2 - \frac{1}{n} & \text{for } \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} \\
1 & \text{for } \frac{1}{2} < t \leq 1 
\end{cases}
\]

This is Cauchy since:
\[
||x_n - x_m|| = \int_{\frac{1}{2}}^{1} |x_n(t) - x_m(t)| \, dt \to 0
\]

But consider that drawn:

\[
x(t) = \begin{cases} 
0 & \text{for } 0 \leq t < \frac{1}{2} \\
1 & \text{for } t \geq \frac{1}{2}
\end{cases}
\]

So the worry is whether our defnition of a space of square integrable functions is complete.

That is, is there any Cauchy sequence of s.i. functions whose limit is not s.i.?

According to Riesz and Fischer, the answer is no, so our space is complete.

Let the functions \( f_1, f_2, \ldots \) be a sequence.

If \( \lim_{n \to \infty} ||f_n - f_m||^2 = \lim_{n \to \infty} \int_0^1 |f_n(x) - f_m(x)|^2 \, dx = 0 \), then there exists a square (Lebesgue) integrable function \( f(x) \) to which the sequence \( f_n(x) \) converges "in the mean."

That is, \( \lim_{n \to \infty} \int_0^1 |f_n(x) - f(x)|^2 \, dx = 0 \) (the proof is h.r.f.)