For Legendre we started on \([-1,1]\) \(w/f\) \((f,g)=\int_{-1}^{1} f(x)g(x)dx\).

Let's consider other options. This "weight" renders finite what might otherwise be infinite over \((-\infty,\infty)\).

\((-\infty,\infty)\) \(w/f\) \((f,g)=\int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx\) non-trivial weight.

Starting \(w/f\) \([x^n, x^1, x^2, ...\) let's Gram-Schmidt it:

\(\hat{P}_0(x) = \frac{1}{\sqrt{\pi}}\) \(\langle \hat{P}_0, \hat{P}_0 \rangle = \int_{-\infty}^{\infty} e^{-x^2}dx = 1\)

\(\hat{P}_1(x) = \frac{x - \hat{P}_0(\hat{P}_0,x)}{\|x - \hat{P}_0(\hat{P}_0,x)\|}\) but \(\langle \hat{P}_0, x \rangle = \left\langle \frac{1}{\sqrt{\pi}}, x \right\rangle \int_{-\infty}^{\infty} xe^{-x^2}dx = 0\)

\(= \frac{x}{\sqrt{\pi}}\) since \(\langle x, x \rangle = \int_{-\infty}^{\infty} xe^{-x^2}dx = \frac{1}{\sqrt{\pi}} \quad \Rightarrow \quad ||x|| = \frac{1}{\sqrt{\pi}}\)

\(\hat{P}_2(x) = \frac{x^2 - \hat{P}_0(\hat{P}_0,x) - \hat{P}_1(x, \hat{P}_1,x)}{\|x^2 - \hat{P}_0(\hat{P}_0,x) - \hat{P}_1(x, \hat{P}_1,x)\|}\) but \(\langle \hat{P}_0, x^2 \rangle = \left\langle \frac{2}{\sqrt{\pi}}, x^2 \right\rangle \) while \(\langle \hat{P}_1, x \rangle = \left\langle \frac{1}{\sqrt{\pi}}, x \right\rangle \int_{-\infty}^{\infty} xe^{-x^2}dx = 0\)

\(= \frac{x^2 - \frac{1}{2}}{\sqrt{\frac{3}{\pi}}(x - \frac{1}{2})}\) since \(\int_{-\infty}^{\infty} xe^{-x^2}dx = \frac{\sqrt{\pi}}{2}\) and others from above.

Once again we can Rodrigues it and find:

\(\hat{P}_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}\) \(\Rightarrow\) \(\hat{P}_0(x) = (-1)^0 e^{\frac{x^2}{2}} e^{-x^2} = 1 \Rightarrow N = (\frac{\sqrt{\pi}}{2})^n\)

These are the "Hermite" polynomials:

\(\hat{P}_0(x) = (-1)^0 e^{\frac{x^2}{2}} e^{-x^2} = 1\) \(\quad N = (\frac{\sqrt{\pi}}{2})^n\)

\(\hat{P}_1(x) = (-1)^1 e^{\frac{x^2}{2}} \frac{d}{dx} e^{-x^2} = -(-x) = \frac{1}{\sqrt{\pi}} x \quad N = (\frac{\sqrt{\pi}}{2})^n\)

\(\hat{P}_2(x) = (-1)^2 e^{\frac{x^2}{2}} \frac{d^2}{dx^2} e^{-x^2} = \frac{1}{\sqrt{\pi}} x^2 + 2x \quad N = (\frac{\sqrt{\pi}}{2})^n\)

\(\hat{P}_3(x) = (-1)^3 e^{\frac{x^2}{2}} \frac{d^3}{dx^3} e^{-x^2} = \left\langle \frac{1}{\sqrt{\pi}}, x^3 \right\rangle \int_{-\infty}^{\infty} e^{-x^2}dx = -\frac{3}{\sqrt{\pi}} x^2 + 2x \quad N = (\frac{\sqrt{\pi}}{2})^n\)

Are these complete? Recall that for Legendre, we utilised Weierstrass, however in this case our "interval" is \((-\infty,\infty)\) and so does not fall under the Weierstrass argument.

Now since the Hermites are polynomials, we know that closure (and hence completeness) can be shown by demonstrating that \((f(x), x^n) = \int_{-\infty}^{\infty} f(x) x^n e^{-x^2}dx = 0 \Rightarrow f(x)\) is almost \(0\) for all \(n\). This can be shown via techniques a bit beyond our aims.
To get the equation satisfied by these consider the following generating function:

$$\Theta(x,t) = e^{xe^{-(t-x)^2}} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

(from which we can derive Rodrigues)

Note: \( \frac{\partial}{\partial x} = 2x e e^{-(t-x)^2} + (t-x) e e^{-(t-x)^2} = 2t\phi \Rightarrow \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n e^{t-x} \)

Equating powers of \( t^n \Rightarrow H_n(x) = \frac{n!}{(n-1)!} H_{n-1}(x) = 2n H_{n-1}(x) \quad \text{RR#1 (recursion relation)} \)

Also note: \( \frac{\partial}{\partial x} = -e^{-(t-x)^2} \phi = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n-1} = -2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n-1} + 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \)

Equating powers of \( t^n \Rightarrow H_n(x) = n! \left[ -2 \frac{H_{n-1}(x)}{(n-1)!} + 2x \frac{H_{n-1}(x)}{n!} \right] = -2x H_{n-1}(x) + 2x H_n(x) \quad \text{RR#2} \)

What we would like is an equation involving only \( H_n(x) \). Combining RR#1 and RR#2 we have:

\[ H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0 \quad \text{RR#2} \]

\[ H_n(x) - 2x H_{n-1}(x) + 2(n-1) H_{n-2}(x) = 0 \]

\[ \frac{H_n(x)}{2n} \quad \text{RR#1} \rightarrow \frac{H_{n+1}(x)}{2(n+1)} = \frac{H_n(x)}{2n} \quad \text{using RR#1 again} \]

\[ H_n(x) - \frac{ax}{2n} H_n(x) + \frac{a^2 n!}{(n-1)!} H_n(x) = 0 \]

\[ 2n \frac{H_n(x)}{2n} - 2x H_n(x) + H_n(x) = 0 \quad \text{for } n \geq 0 \quad \text{Hermite's equation} \]

You get to do Laguerre in your HW!
Lecture 14 - Legendre, Laguerre and Hermite walk into a bar...

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We have arrived at the following alternative means of defining Legendre, Hermite and Laguerre polynomials, they are solutions of

\[(x^2-1)\frac{d^2}{dx^2}P_n(x) + 2x \frac{d}{dx}P_n(x) - n(n+1)P_n(x) = 0 \quad x \in [-1, 1] \quad (f, g) = \int_{-1}^{1} f g \, dx \quad \text{Legendre}\]
\[H_n(x) - 2xH_n(x) + nh_n(x) = 0 \quad x \in (-\infty, \infty) \quad (f, g) = \int_{-\infty}^{\infty} f g e^{-x^2} \, dx \quad \text{Hermite}\]
\[xL_n(x) + (1-x)L_n(x) + nL_n(x) = 0 \quad x \in [0, \infty) \quad (f, g) = \int_{0}^{\infty} f g e^{-x} \, dx \quad \text{Laguerre}\]

All of these can be promoted to \((-\infty, \infty)\) intervals using:

In general \(\omega > 0\)

\[\int_{-\infty}^{\infty} f g \omega(x) \, dx = \int_{-\infty}^{\infty} f g e^{-\omega x} \, dx\]

\[\text{Legendre: } w(x) = 1, \quad \text{Hermite: } w(x) = e^{-x^2}\]

All of these can be cast as:

\[L_n = \lambda_n w(x) \quad L = \lambda(x) \frac{d^2}{dx^2} + \alpha(x) \frac{d}{dx} + \gamma(x) \quad \text{and } \lambda \text{ a constant } \left(\alpha, \beta, \gamma \text{ are real}\right)\]

An important feature of this story is that the \(L_n\)’s are Hermitean w.r.t. the inner-products.

Let’s explore this in general. First of all, our definition of Hermite:

\[L \text{ is Hermite if: } (Lf, g) = \int_{-\infty}^{\infty} f \omega w(x) \, dx = \int_{-\infty}^{\infty} \omega \frac{\partial^2}{\partial x^2} f g \, dx = (f, Lg)\]

So for \(L\) to be Hermite we need the r.h.s. = l.h.s. = 0,

r.h.s. \( (f, Lg) = \int_{-\infty}^{\infty} f \int_{-\infty}^{\infty} \omega g + \Delta(x) \frac{d}{dx} + \gamma(x) \, dx \)

l.h.s. \( (Lf, g) = \text{same w/ } f \leftrightarrow g \text{ switched} \)

Then: r.h.s. - l.h.s. = 0 \Rightarrow \int_{-\infty}^{\infty} \omega \left( f \frac{\partial}{\partial x} - g \frac{\partial}{\partial x} \right) \, dx = 0 \Rightarrow L_{\omega} = \text{linear operator}

For \(L\) to be Hermite, this must be true for arbitrary \(f\) and \(g\) (which means \(f \frac{\partial}{\partial x} - g \frac{\partial}{\partial x} \cdot 0\)) so:

1. \(L_{\omega} = 0 \Rightarrow \omega = \text{either } \omega(x) = \text{constant, or the functions } f, g \text{ have } L\)-smooth \(\frac{\partial}{\partial x} \omega(x) \)

2. \(L_{\omega(x)} = \omega(x) \Rightarrow (\omega(x)) \frac{d}{dx} \omega(x) = \omega(x) \Rightarrow \) \(C = 0\) is a solution,

\[\int_{-\infty}^{\infty} \omega(x) \, dx = \begin{cases} \infty & x > 0 \text{ or } x < 0 \text{ if } a > 0, \quad a < 0 \text{ if } C > 0 \end{cases}\]
Posing this requirement of Hermiteness on the original eqn. we can rewrite:
\[ Lu = u^{(n)} + Bu + \gamma u = \lambda u \]

as
\[ \frac{d}{dx} \left[ w(x) \frac{du}{dx} \right] + (x-\lambda)w(x)u = 0 \]
\[ (w(x)u')' + w(x)u'' + \gamma w(x)u = \gamma \delta u' + \gamma \delta u'' + \gamma w(x)u = \gamma u \]

which together with \[ \int_0^\infty \left[ w(x)f_1(x) - f_2(x) \right] e^{-\lambda x} \, dx = 0 \]
define a Sturm–Liouville system.

Now recall that in the finite-dimensional case we had that:
\[
\text{If } A \text{ is normal, then the eigenvectors belonging to distinct eigenvalues are orthogonal.}
\]

Well it turns out that at least for Hermitian \( A \), this result extends to infinite-dimensional \( A \).

Proof: \( \lambda_n \) and \( \lambda_m \) w.m.t. \( \frac{d}{dx} \left[ (w(x)) \frac{du}{dx} \right] + (x-\lambda)w(x)u = 0 \)
Recall that for just one of the other: \( \frac{d}{dx} \left[ (w(x)) \frac{du}{dx} \right] + (x-\lambda)w(x)u = 0 \)
Hence: \( \lambda_m \neq \lambda_n \), \( \lambda_m \neq \lambda_n \).

Then:
\[ \frac{d}{dx} \left[ (w(x)) \frac{du}{dx} \right] - \lambda_m w(x)u - \lambda_n \frac{d}{dx} \left[ (w(x)) \frac{du}{dx} \right] + \lambda_n w(x)u = 0 \]
\[ \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} u_m - u_m \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] \right] = - (\lambda_m - \lambda_n) w(x)u \]
Since \( \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} u_m - u_m \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] \right] = \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] - \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] \frac{d}{dx} u_m + u_m \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] \]
then \( \int_0^\infty \frac{d}{dx} u_m \, dx \) both sides
\[ \left[ \frac{u_n(w(x))}{u_m(w(x))} u_m - u_m \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] \right] \int_0^\infty w(x)u_m \, dx = - (\lambda_m - \lambda_n) \int_0^\infty w(x)u \, dx = - (\lambda_m - \lambda_n) (u_m, u_n) \]
\[ \left[ w(x) \left( \frac{u_n(w(x))}{u_m(w(x))} u_m - u_m \frac{d}{dx} \left[ \frac{u_n(w(x))}{u_m(w(x))} \right] \right) \right] \int_0^\infty w(x)u_m \, dx = 0 \] by condition for S.L.

So we have \( (\lambda_m - \lambda_n) (u_n, u_m) = 0 \) but \( \lambda_m \neq \lambda_n \) \( \Rightarrow (u_n, u_n) = 0 \).
Now believe it or not, everything we have said about S.I. systems so far has not been restricted to polynomial functions.

Now let's explore making this restriction.

\[ L \cdot Q_n = \lambda \cdot Q_n \quad \text{with} \quad L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c \quad \text{and} \quad (Q_n, Q_m) = 0 \quad \text{for} \quad n \neq m \]

\[ L = \text{must be polynomial in} \quad \alpha, \beta, \gamma. \quad \text{In order not change the degree of} \quad Q_n \quad \text{we need:} \]

\[ \alpha(x) = a_0 x^d + a_1 x + a_2 \]

\[ \beta(x) = b_0 x + b_1 \]

\[ \gamma(x) = c_0 \]

An interesting observation is that \( \beta(x) \) can never be zero, because if it were then \( (\infty) = \beta(0) = 0 \quad \text{or} \quad \beta(x) \quad \text{constant} \), but then \( \int_{-\infty}^{\infty} \beta(x) f(x) g(x) dx \rightarrow 0 \) since both \( f \) and \( g \) being polynomial mean they do not vanish at \( \pm \infty \).

In fact, we must grow faster than any inverse power of \( x \) in order to beat all \( f \) and \( g \).

Now it seems that six parameters \( \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \gamma_0 \) are the freedom in defining a S.I. system.

It's actually less than that due to:

1. \( L \) may be scaled by \( C_1 \), constant \( \Rightarrow \) \( L \cdot C_1 = Q_n \) unchanged, \( \lambda = \lambda \).

2. \( x \) may be shifted by \( C_2 \), constant \( \Rightarrow \) \( Q_n(x + c) = Q_n(x) \) and \( \lambda = \lambda \).

3. \( x \) may be scaled by \( C_3 \), constant \( \Rightarrow \) \( Q_n(Cx) = Q_n(x) \) and \( \lambda = \lambda \).

4. \( L \) may be shifted by \( C_4 \), constant \( \Rightarrow \) \( L + C_4 = Q_n \) unchanged and \( \lambda = \lambda + C_4 \).

The last one means that we can in general set \( \gamma_0 = 0 \). That leaves five parameters, but owing to 1-3, we can expect there to be only two independent parameters.

So we will organize and analyze things by the order of \( \alpha_0 \):

1. \( \alpha(x) \) is quadratic \( (\alpha_0 \neq 0) \) \[ \begin{align*}
&\{ \text{If not specified, coefficient can be zero or not!} \}
\end{align*} \]

2. \( \alpha(x) \) is linear \( (\alpha_0 = 0, \alpha_1 \neq 0) \)

3. \( \alpha(x) \) is constant \( (\alpha_0 = \alpha_1 = 0, \alpha_2 \neq 0) \)
\( \alpha(x) \) is quadratic.

First let's choose \( C \), s.t. \( C L \Rightarrow \alpha_0 = 1 \) then \( \omega x = C e^{\frac{L}{2}} = C \exp \left[ \frac{Lx + \alpha}{x^2 + \alpha x + \alpha} \right] \)

Now \( \alpha(x) = x^2 + \alpha, x + \alpha \) is composed of real coefficients, but the roots of \( \alpha(x) \) could be complex.

For complex roots: \( \alpha(x) = (x - \rho)(x - \rho^*) \) and \( \omega x = C \exp \left[ \frac{(x - \rho)(x - \rho^*)}{(x - \rho)(x - \rho^*)} \right] \)

Since \( \omega x \) is a polynomial and we need \( \omega(x) - q^2 - q^* \) to be zero, there are obviously polynomials for \( q \) with powers such that \( \omega(x) \) doesn't vanish fast enough.

Also \( \omega(x) \) is never zero (since \( \exp(x+itx) \to 0 \) and the roots are complex so only 0 for values of \( x \) off the real line).

Thus complex roots won't work.

For real roots: Using \( C_1, C_2, C_3 \) we can make \( \alpha(x) = 1 - x^2 \)

\[
\begin{align*}
C_1: & \quad L \Rightarrow \frac{4\pi}{\sqrt{1 - x^2}} \\
C_2: & \quad x \Rightarrow \frac{1}{1 - x^2} \\
C_3: & \quad x \Rightarrow \frac{x}{\sqrt{1 - x^2}}
\end{align*}
\]

\[
\frac{\omega x}{\sqrt{1 - x^2}} \Rightarrow \frac{\int (1 - x^2)(x^2 + \alpha x + \alpha) dx}{\sqrt{1 - x^2}} = \frac{4\pi}{\sqrt{1 - x^2}} \left[ 1 - x^2 + \frac{x^2}{2} \sqrt{1 - x^2} x + \alpha \right]
\]

\[
C_2: \quad x \Rightarrow x - \frac{\alpha}{\sqrt{1 - x^2}}
\]

\[
\int (1 - x^2)(x^2 + \alpha x + \alpha) dx = \frac{4\pi}{\sqrt{1 - x^2}} \left[ 1 - x^2 + \frac{x^2}{2} \sqrt{1 - x^2} x + \alpha \right]
\]

\[
= -x^4 + 4
\]

We can also use \( \beta = q - \rho, \delta = -(p + q + \rho) \) (just a redefinition) and then things clean up nicely:

\[
\frac{\omega x}{\sqrt{1 - x^2}} = \int \frac{1 - x^2}{1 - x^2} = \frac{q + 1}{1 - x^2} \Rightarrow \omega x = C e^{\frac{L}{2}} = C \frac{1 + x}{1 - x^2}
\]

\[
\Rightarrow \omega(x) = C \left( 1 + x \right)^{q/2} \left( 1 - x \right)^{q/2}
\]

Now, once again we want to go to zero fast enough to beat higher powers of \( x \) in \( \omega(x) \).

But the roots of \( \alpha(x) = 1 - x^2 \) allow us to use the nonzero part between those and glue it to \( 0 \) for \( x < 1 \). \( x > 1 \) (the boxes). So we are led to \( C(-1, 1) \) though this can easily be shifted to any \( C(a, b) \) by adjusting \( C_1, C_2, C_3 \).
So in the end \([-1,1]\) \(w(x) = (1+x)^p (1-x)^q\) works and gives rise to the \(\text{Jacobi polynomials} f_n(x)\) where \(p, q\) refer to the values in \(w(x)\). That is, you pick \(p\) and \(q\) and using \(w(x)\) find an entire set of polynomials specific to these \(p\) and \(q\).

**Special cases:**

If \(p=q=m>1\) \(w(x) = (-x^2)^m\) \(\Rightarrow\) \(\text{Gegenbauer polynomials} \ G_n^{\pm m}(x) = J_n^{\pm m}(x)\).

If \(p=q=-\frac{1}{2}\) \(w(x) = (1-x)^{\frac{-1}{2}}\) \(\Rightarrow\) \(\text{Chebyshev polynomials} \ \bar{T}_n(x) = J_{\frac{1}{2}}(x)\).

If \(p=q=0\) \(w(x) = 1\) \(\Rightarrow\) \(\text{Legendre polynomials} \ \bar{P}_n(x) = J_0(x)\).

**\(\alpha(x)\) is linear:**

\[\alpha(x) = x + \alpha_2 \quad (b_2 \leq C_1) \Rightarrow \alpha(0) = 0 \text{ at } x=0\]

Then:

\[w(x) = C \exp \left[ \int \frac{dx}{x} \right] = C x^{\frac{1}{2}} e^{\frac{1}{x}}\]

If 0 < \(x\) then \(w(x)\) goes to 0 fast enough \(\Rightarrow\) \(x \to \infty\) for any power of \(x\) in \(f(x)\), but of course as \(x \to -\infty\) then \(w(x)\) \(\to 0\).

So we use the root at \(x=0\) to pair it up with 0 for all \(x<0\) (Hessians).

Using \(C_2\) to set \(B_0 = -1\) and calling \(B_s = s+1\) \(\Rightarrow\) \(s>1\) then \(w(x) = x^se^{-x}\) for \(C_0,\infty\).

For \(s=0\) these are the \text{Laguerre polynomials} \(L_n(x) = \bar{L}_n(x)\).

For \(s>0\) they are the \text{associated Laguerre polynomials} \(L_s^n(x)\).
\( a(x) \text{ is constant} \)

\[ c_i \Rightarrow a_i = 1 \Rightarrow a(x) = 1 \Rightarrow \omega = \exp \left[ \int (\delta_0 x + \delta_1) \, dx \right] \]

\[ = \exp \left[ \frac{\delta_1}{2} x^2 + \delta_0 x \right] \]

\[ = \exp \left( -\frac{\delta_1}{2\delta_0} \right) \exp \left[ \frac{\delta_1}{4} (x + \frac{\delta_0}{\delta_1})^2 \right] \]

If \( \delta_0 < 0 \) then \( \omega = \exp \) falls exponentially as \( x \to \infty \) so it works!

With \( c_i \) and \( c_j \) we have \( \delta_0 = -2, \delta_1 = 0 \) and \( c' = 1 \Rightarrow \omega(x) = e^{-x^2/2} \) on \( (-\infty, \infty) \)

In this case \( Q_n(x) \) are the Hermite.
Various properties

Eigenvalues: \( \lambda_n = \lambda_n Q_n \Rightarrow (\alpha x^2 + \beta x + \gamma) \frac{d^2}{dx^2} Q_n + (\beta_0 x + \beta_1) \frac{d}{dx} Q_n = \lambda_n Q_n \)

All \( x^k \) terms must have coefficients that satisfy this, so \( \alpha_n (n-1) + \beta_n = \lambda_n \rightarrow n (\alpha_0 n + \beta_0 - \gamma_0) \)

Thus \( \lambda_n \) are spaced linear in \( n \) for \( \gamma_0 = 0 \) cases (Legendre and Laguerre) and quadratically for \( \gamma_0 \neq 0 \) or Jacobi.

Rodriguez: Recall \( P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - 1)^n \) Legendre w/ \( w(x) = 1 \)

\( L_n(x) = (-1)^n e^x \frac{d^n}{dx^n} e^{-x} \) Hermite w/ \( w(x) = e^{-x} \)

\( L_n(x) = \frac{2^n}{n!} \frac{d^n}{dx^n} (e^{-x}) \) Laguerre w/ \( w(x) = e^{-x} \)

These can all be generated from: \( Q_n(x) = \left( \frac{\alpha_n}{\beta_n} \right) \frac{d^n}{dx^n} \left( \alpha e^{-x} \right) \)

Completeness:

The orthonormal set of Sturm-Liouville polynomials \( \{Q_n(x)\} \) is complete in Hilbert space.

To prove it we can show they are closed, hence complete.

To prove closed we need to show that if \( (f, Q_n) = 0 \) for all \( n \Rightarrow f \) is almost 0.

Since \( Q_n \) are polynomials in \( x \), then \( x^n \) can be written as linear combo of \( Q_n(x) \).

That is if \( (f, Q_n) = 0 \Rightarrow (f, x^n) = 0 \) for all \( n \).

Consider: \( q(k) = \int_0^\infty f(x) e^{-ikx} \) w/ real \( k \)

Expanding: \( e^{-ikx} = \sum_{n=0}^{\infty} \frac{(-ikx)^n}{n!} x^n \) and using \( (x^n, f) = 0 \Rightarrow q(k) = 0 \) for all \( k \).

But an argument to come later implies that the above implies \( f(x) w(x) = 0 \) almost everywhere.

\( \Rightarrow \) whenever \( w(x) \neq 0 \), \( f(x) = 0 \) almost everywhere \( \Rightarrow \) St. pols are closed, hence complete.