So here we are going to capitalize on your exposure to ideas in QM with a formal definition of it. We will also carefully migrate from the simple story of finite-dim. vector spaces to the in-finite-dim. cases.

In the finite case a very useful idea is the "resolution of the identity in terms of projection operators." Imagine an N-dim. vector space and a Hermitian operator \( A \) acting on it. Suppose \( A \phi_n = \lambda_n \phi_n \) where \( \{ \phi_n \} \) is the set of eigenvectors associated with \( A \). Assume \( \lambda_n \) are non-deg.

Then: \[ I \phi = \sum_{n=1}^{N} e_n (\phi, \phi) \] \( \therefore \) if \( \phi = e_a + \phi_c \) then: \[ I \phi = \phi (\phi, \phi) = \phi_c + \phi_c + \phi \]

\[ P_n \phi = \phi_n (\phi_n, \phi) = \phi_n + \phi_n \Rightarrow P_n^2 = P_n \phi_n = \phi_n \] \( \phi_n \) projection

There: \[ I = \sum_{n=1}^{N} P_n \]

And: \[ A = \sum_{n=1}^{N} \lambda_n P_n \] since \( A \phi_n = \sum_{n=1}^{N} \lambda_n P_n \phi_n = \lambda_n \phi_n \) and: \( P_n \phi_n = 0 \) non-

Going to \( \infty \) will require some rethinking. So once again, for finite-dim. consider:

\[ E(\lambda) = \begin{cases} \lambda, & \lambda < \lambda_1 \\ \sum_{n=1}^{N} P_n, & \lambda_1 < \lambda < \lambda_{N-1} \\ \sum_{n=1}^{N} P_n, & \lambda \geq \lambda_N \end{cases} \Rightarrow \begin{cases} E(-\infty) = 0, \quad E(\infty) = I \\ \lambda \in \lambda_1 \lambda_2 \quad \text{then} \quad E(n)E(n) = \sum_{n=1}^{N} P_n \sum_{n=1}^{N} P_n = E \sum_{n=1}^{N} P_n \end{cases} \]

\[ E(\lambda) = \begin{cases} \lambda, & \lambda < \lambda_1 \\ \sum_{n=1}^{N} P_n, & \lambda_1 < \lambda < \lambda_{N-1} \\ \sum_{n=1}^{N} P_n, & \lambda \geq \lambda_N \end{cases} \]

Now it turns out that by using the Stoltzfus form of integrals: \[ \int_{a}^{b} f(x) d\lambda (x) = \lim_{N \to \infty} \sum_{n=1}^{N} \int_{x_{n}}^{x_{n+1}} f(x) d\lambda (x) \]

where \( \lambda \) is some point between \( x_i \) and \( x_{i+1} \), then:

\[ \int_{a}^{b} d E(\lambda) = \sum_{n=1}^{N} P_n \] which basically remains constant as \( \lambda \) increases until we hit an eigenvalue and gain a \( P_n \), so:

\[ I = \int_{a}^{b} d E(\lambda) \]

\[ A = \int_{a}^{b} \lambda d E(\lambda) \]
We are now in a position to generalize to arbitrary Hilbert space $H$:

For any self-adjoint $A$ on $H$, there exists a unique operator valued $E(\lambda)$ s.t.

1. $E(\lambda_a)E(\lambda_b) = E(\lambda_a \cdot \lambda_b)$ for $\lambda_a \leq \lambda_b$
2. $\lambda \uparrow \infty E(\lambda) = 0$, $\lambda \downarrow -\infty E(\lambda) = I$
3. $I = \int_\infty^{-\infty} E(\lambda) d\lambda$
4. $A = \int_\infty^{-\infty} \lambda dE(\lambda)$

$E(\lambda)$ is the resolution of $I$ for $A$.

The points $\lambda$ for which $E(\lambda)$ is not constant is the "spectrum" of $A$. 

$[E(\lambda), A] = 0$, $\ldots$ s.t. $[\lambda A, B] = 0 \Rightarrow [E(\lambda_1), B] = 0$
Now we are ready: (A for Action)

[A] Any physical system is completely described by a normalized vector in $\mathcal{H}$.

[B] To every physical observable there corresponds a self-adjoint $A$ on $\mathcal{H}$.

Sometimes we can build them from classical expressions $m\omega^2\dot{\varphi} + \frac{1}{2}m\dot{\varphi}^2$, $p\rightarrow i k \varphi$, but sometimes (spin) not!

[C] Physical measurement of observable $A$ are elements of the spectrum of $A$.

Since $A$ is s.a., its spectrum is real. Good for the real world we live in!

To see and appreciate the need for $E(\lambda)$, consider $A = \hat{p} = -i \hbar \frac{d}{dx}$

We could try and say $A \phi_k = \lambda_k \phi_k$ for $\phi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$, the $\phi_k$ are eigenfunctions which are normalized by $\int_{-\infty}^{\infty} \phi_k^* (x) \phi_{k'} (x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{ik'x} dx = \delta (k - k')$

Now let's define $\hat{p} \phi = \phi_k (\phi_k, \phi)$ for $\phi \in \mathcal{H}$. Sounds good right? No! $\phi_k \notin \mathcal{H}$.

Instead let's define $E(k) \phi = \frac{i}{\hbar} \sum_{k'} e^{-i(k'-k)\lambda} \int_{-\infty}^{\infty} \phi_k^* (x) \phi \left( x + \frac{i}{\hbar} (k'-k) \right) dx$.

Is $E(k)$ doing what it should?

First: $E(-\lambda) = 0$ since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ikx} e^{-i\lambda x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\lambda} \delta (k + \lambda) \phi_k^* \phi \phi = f(x), \int_{-\infty}^{\infty} \phi \phi dx = \mathbb{I}$

What about $E(k_1) E(k_2) = E(k_1)$ for $k_1 < k_2$?

$E(k_1) E(k_2) = E(k_1) \frac{i}{\hbar} \sum_{k'} e^{-i(k'-k_1)\lambda} \int_{-\infty}^{\infty} \phi_k^* (x) \phi \left( x + \frac{i}{\hbar} (k'-k_1) \right) dx$.

When $\phi_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(k'-k)\lambda} \int_{-\infty}^{\infty} \phi_k^* (x) \phi \left( x + \frac{i}{\hbar} (k'-k) \right) dx$.

Since $k_1 < k_2$, $E(k_1) E(k_2) = \frac{i}{\hbar} \sum_{k'} e^{-i(k'-k_1)\lambda} \int_{-\infty}^{\infty} \phi_k^* (x) \phi \left( x + \frac{i}{\hbar} (k'-k_1) \right) dx = E(k_1)$.

Lastly, does this $E(k)$ belong to $\hat{p}$? i.e. $\hat{p}_k = \frac{d}{dx} \int_{-\infty}^{\infty} \phi_k^* (x) \phi (x) dx$

$\left[ \int_{-\infty}^{\infty} \phi_k^* (x) \phi (x) dx \right] \left( \frac{d}{dx} \int_{-\infty}^{\infty} \phi_k^* (x) \phi (x) dx \right) = \frac{i}{\hbar} \sum_{k'} e^{-i(k'-k)\lambda} \int_{-\infty}^{\infty} \phi_k^* \phi \left( x + \frac{i}{\hbar} (k'-k) \right) dx$

$= \frac{i}{\hbar} \sum_{k'} e^{-i(k'-k)\lambda} \int_{-\infty}^{\infty} \phi_k^* (x) \phi (x + \frac{i}{\hbar} (k'-k)) dx$

$= -\frac{i}{\hbar} \int_{-\infty}^{\infty} \phi_k^* (x) \phi (x) dx - i \frac{d}{dx} \int_{-\infty}^{\infty} \phi_k (x) \phi (x) dx = \hat{p} \phi (x)$
and now something central:

\[ \text{AIV} \]

Make a measurement of observable \( A \) on a state \( \phi \). The probability that observed \( \lambda \) will lie between \( \lambda_1 \) and \( \lambda_2 \) \((\lambda_1 < \lambda_2)\) is:

\[ P(\lambda_1, \lambda_2) = \| [E(\lambda_2) - E(\lambda_1)] \phi \|^2 \]

where \( E(\lambda) \) is the resolution of \( I \) for \( A \).

Two extremes:

Clearly, state \( |0\rangle \) is:

\[ P(-\infty, \infty) = \| [E(-\infty) - E(\infty)] \phi \|^2 = \| [0 - 1] \phi \|^2 = 1 \]

and if \( E(\lambda) \) is constant between \( \lambda_1 \) and \( \lambda_2 \):

\[ P(\lambda_1, \lambda_2) = \| [E(\lambda_2) - E(\lambda_1)] \phi \|^2 = \| 0 \|^2 = 0 \]

The in between:

If \( \lambda_1 < \lambda < \lambda_2 \) and \( \lambda \) is a part of the spectrum of \( A \), then:

\[ A [E(\lambda_2) - E(\lambda_1)] \phi = \int_{\lambda_1}^{\lambda_2} \lambda \; d \lambda \; [E(\lambda) - E(\lambda_1)] \phi \]

\[ = \int_{\lambda_1}^{\lambda_2} \lambda \; d \lambda \; [E(\lambda) E(\lambda_1) \phi] - \int_{\lambda_1}^{\lambda_2} \lambda \; d \lambda \; [E(\lambda) E(\lambda_2) \phi] \]

\[ = \int_{\lambda_1}^{\lambda_2} \lambda \; d \lambda \; [E(\lambda) \phi] - \int_{\lambda_1}^{\lambda_2} \lambda \; d \lambda \; [E(\lambda_1) \phi] \]

\[ = \int_{\lambda_1}^{\lambda_2} \lambda \; d \lambda \; [E(\lambda) - E(\lambda_1)] \phi \]

\[ \Rightarrow \lambda \phi \Lambda = A \phi \Lambda = \lambda \phi \Lambda \]

Now if \( \phi_k \) is a unique normalized eigenvector associated with \( \lambda_0 \), then:

\[ [E(\lambda_2) - E(\lambda_1)] \phi = \phi_k (\phi_k, \phi) = 0 \]

then we can write:

\[ P(\lambda_1, \lambda_2) = |(\phi_k, \phi)|^2 \]

Note: For \( A = p_x \), this gives:

\[ P(\kappa_1, \kappa_2) = \| [E(k_2) - E(k_1)] \phi \|^2 \]

\[ = \| \int_{k_1}^{k_2} \phi_{k'} (\phi_{k'}, \phi) dk' - \int_{k_1}^{k_2} \phi_{k'} (\phi_{k'}, \phi) dk' \|^2 \]

\[ = \| \int_{k_1}^{k_2} \phi_{k'} (\phi_{k'}, \phi) dk' \|^2 \]

\[ = \| \frac{1}{\sqrt{\pi}} \int_{\kappa_1}^{\kappa_2} \left( \int_{\gamma}^{\gamma + i\kappa'} \phi (k') \hat{\phi} (k') dk' \right) \hat{\phi} (\gamma) d\gamma \|^2 \]

\[ = \| \frac{1}{\sqrt{\pi}} \int_{\kappa_1}^{\kappa_2} e^{i k' \cdot x} \hat{\phi} (k') dk' \|^2 \]

\[ = \frac{1}{\pi} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} \hat{\phi} (k') \hat{\phi} (k'') \cdot e^{i (k' - k'') \cdot x} dk' dk'' \]

\[ = \frac{1}{\pi} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_1}^{\kappa_2} e^{i (k' - k'') \cdot x} \hat{\phi} (k') \hat{\phi} (k'') \cdot \hat{\phi} (k'') \cdot \hat{\phi} (k') \cdot d\gamma d\gamma \]

\[ = \int_{\kappa_1}^{\kappa_2} |\hat{\phi}(k')|^2 dk' \]