We have arrived at the following alternative means of defining Legendre, Hermite and Laguerre polynomials, they are solutions of

\[ (x^2-1)\frac{d^2}{dx^2}P_n(x) + 2x\frac{d}{dx}P_n(x) - n(n+1)P_n(x) = 0 \quad x \in [-1,1] \quad (f,g) = \int_{-1}^{1} f(x)g(x)\,dx \quad \text{Legendre} \]

\[ H_n(x) - 2nH_{n-1}(x) + n(n+1)H_n(x) = 0 \quad x \in (-\infty,\infty) \quad (f,g) = \int_{-\infty}^{\infty} \frac{e^{-x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x} \,dx \quad \text{Hermite} \]

\[ xL_n(x) + (1-x)L_n(x) \quad n \leq L_n(x) = 0 \quad x \in [0,\infty) \quad (f,g) = \int_{0}^{\infty} e^{-x} \,dx \quad \text{Laguerre} \]

All of these can be promoted to \((-\infty,\infty)\) intervals using:

\[ \text{In general } w(x) \in \left\{ \begin{array}{l}
\text{i) Nothing on Hermite (or rather } w(x) = e^{-x} \\
\text{ii) Heaviside for Legendre } w(x) = H(x) \\
\text{iii) Boxcar for Legendre } w(x) = \text{boxcar}(x) = H(x+1) - H(x-1)
\end{array} \right. \]

All of these can be cast as:

\[ L = \text{Heaviside operator } \frac{d^{2}}{dx^{2}} + \Delta(x) \frac{d}{dx} + \gamma(x) \quad \text{and } \lambda \text{ a constant (all } \Delta,\gamma,\lambda \text{ are real).} \]

An important feature of this story is that the \(L\)'s are Hermitean w.r.t. the inner-products.

Let's explore this in general. First of all, our definition of Hermitean:

\[ L \text{ is Hermitean if } (Lf,g) = \int_{-\infty}^{\infty} (Lf)g(x)\,dx = \int_{-\infty}^{\infty} f(x)g(x)\,dx = (f,Lg) \]

So for \(L\) to be Hermitean we need the r.h.s. - l.h.s. = 0.

r.h.s. \[ (f,Lg) = \int_{-\infty}^{\infty} f(x)[(Af'' + Bf' + Cf)g(x) - Ce^{x}g(x)]\,dx \]

\[ = \left[ \int_{-\infty}^{\infty} f(x)g(x)\,dx \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (f(x)g(x))\,dx \]

l.h.s. \[ (Lf,g) = \text{same } w/ \text{ } f \leftrightarrow g \text{ switched} \]

Then: r.h.s. - l.h.s. = 0 \[ \Rightarrow \left[ \text{w/} f \leftrightarrow g \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (f(x)g(x))\,dx = 0 \]

Note that 1st, 3rd and 5th terms cancel!

For \(L\) to be Hermitean, this must be true for arbitrary \(f\) and \(g\) (which means \(f,g \rightarrow f\rightarrow f,0\)) so:

1. \[ \left[ \text{w/} f \leftrightarrow g \right]_{-\infty}^{\infty} = 0 \] either \(w=0\) at \(x \rightarrow \infty\), or the functions \(f,g \rightarrow 0\) as \(x \rightarrow \infty\)

2. \[ (w(x))^2 = w(x) = 0 \quad \text{w/ } x \rightarrow \infty \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)g(x)\,dx = 0 \]

\[ \Rightarrow \left\{ \begin{array}{l}
\text{If } w > 0, \text{ then } C = 0 \\
\text{If } w < 0, \text{ then } C = 0 \end{array} \right. \]
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Pursuing this requirement of Hermiticity in the original eqn, we can rewrite:

\[ Lu = \alpha u'' + \beta u' + \gamma u = \lambda u \]

as

\[ \frac{d}{dx} \left[ w(x) \frac{du}{dx} \right] + (\gamma - \lambda) w u = 0 \quad \Rightarrow \quad (w(x)u' + w(x)u)'' = \gamma w u' + \gamma w u = \gamma w u \]

which together with \( [w(x)f^2 - f^2]_{-\infty}^\infty = 0 \) defines a Sturm-Liouville system.

Now recall that in the finite-dimensional case we had that:

If A is normal, then the eigenvectors belonging to distinct eigenvalues are orthogonal.

Well it turns out that at least for Hermitian A, this result extends to infinite-dimensional at least for S.L. systems.

Proof: \( \lambda_n \) and \( \lambda_k \) w.m. \( \alpha \) \( \frac{d}{dx} \left[ (w(x) \frac{du}{dx} \right] + (\gamma - \lambda) w u = 0 \quad \text{Recall that for just one of them:} \quad \frac{d}{dx} \left[ (w(x) \frac{du}{dx} \right] + (\gamma - \lambda) w u = 0 \quad \text{Hecke, } (\gamma - \lambda) = \text{ real} \]

Then:

\[ u_n \frac{d}{dx} \left[ (w(x) \frac{du}{dx} \right] - \lambda_n w u_n \frac{du}{dx} - u_n \frac{d}{dx} \left[ (w(x) \frac{du}{dx} \right] + \lambda_n w u_n u_n = 0 \]

or

\[ \frac{d}{dx} \left[ u_n (w(x)) \frac{du}{dx} - u_n (w(x)) \frac{du}{dx} \right] = -(\lambda_\alpha - \lambda_n) w u_n u_n \]

then \( \int_{-\infty}^{\infty} dx \) both sides:

\[ \left[ u_n (w(x)) \frac{du}{dx} - u_n (w(x)) \frac{du}{dx} \right]_{-\infty}^{\infty} = -(\lambda_\alpha - \lambda_n) \int_{-\infty}^{\infty} w u_n u_n dx = -(\lambda_\alpha - \lambda_n) (u_n, u_n) \]

\[ [(w(x) \frac{du}{dx} - u_n w(x))_{-\infty}^{\infty} = 0 \quad \text{by condition for S.L.} \]

So we have \( (\lambda_\alpha - \lambda_n)(u_n, u_n) = 0 \) but \( \lambda_\alpha \neq \lambda_n \Rightarrow (u_n, u_n) = 0 \)
New believe it or not, everything we have said about S.L. systems so far has not been restricted to polynomial functions.

Now let's explore making this restriction.

\[ L \ Q_n = \lambda \ Q_n \quad \text{with} \quad L = a \ \frac{d^2}{dx^2} + b \ \frac{d}{dx} + c \quad \text{and} \quad (Q_n, Q_m) = 0 \text{ for } n \neq m \]

\text{poly} \quad \text{poly} \quad \text{poly}

\( L \) must then also be polynomial in \( \alpha, \beta, \gamma \). In order not change the degree of \( Q_n \) we need:

\[ \alpha(x) = \alpha_0 \ x^2 + \alpha_1 \ x + \alpha_2 \]

\[ \beta(x) = \beta_0 \ x + \beta_1 \]

\[ \gamma(x) = \gamma_0 \]

An interesting observation is that \( \beta(x) \) can never be zero, because if it were then

\[ \gamma(x) = 0 \quad \Rightarrow \quad \gamma(x) = \text{constant} \quad \text{but then} \quad \int \gamma(x) \ dx \quad \text{is not polynomials in} \quad x \]

\[ \text{being polynomial means they do not vanish at } \pm \infty. \]

In fact, we must go faster than any inverse power of \( x \) in order to beat all \( f \) and \( g \).

Now it seems that six parameters \( \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \gamma_0 \) are the freedom in defining a S.L. system.

It's actually less than that due to:

1. \( L \) may be scaled by \( C_1 = \text{constant} \Rightarrow L \ Q_n = C_1 \ Q_n \text{unchanged} \quad \lambda \to C_1 \lambda \)

2. \( x \) may be shifted by \( C_2 = \text{constant} \Rightarrow Q_n(x + C_2) = Q_n(x) \text{unchanged} \quad \lambda \to \lambda \)

3. \( L \) may be scaled by \( C_3 = \text{constant} \Rightarrow Q_n(x) = C_3 \ Q_n(x) \text{unchanged} \quad \lambda \to \lambda \)

4. \( L \) may be shifted by \( C_4 = \text{constant} \Rightarrow L + C_4 \Rightarrow Q_n \text{unchanged} \quad \lambda \to \lambda + C_4 \)

The last one means that we can set \( \gamma_0 = 0 \). That leaves five parameters, but owing to 1-3, we can expect there to be only two independent parameters.

So we will organize and analyze things by the order of \( \alpha \).

1. \( \alpha(x) \) is quadratic \( (\alpha_0 \neq 0) \)

2. \( \alpha(x) \) is linear \( (\alpha_0 = 0, \alpha_1 \neq 0) \)

3. \( \alpha(x) \) is constant \( (\alpha_0 = \alpha_1 = 0, \alpha_2 \neq 0) \)
\[ \alpha(x) \text{ is quadratic.} \]

First let's choose \( C \), i.e. \( C, L \Rightarrow \alpha_0 = 1 \) then \( w_0(x) = \exp \left[ \int \frac{4x+4}{x^2+ax+4a} \, dx \right] \)

Now \( \alpha(x) = x^2 + \alpha, x + \alpha \) is composed of real coefficients, but the roots of \( \alpha(x) \) could be complex.

For complex roots: \( \alpha(x) = (x - \alpha_1)(x - \alpha_2) \) and \( w_0(x) = \exp \left[ \int \frac{\alpha_1+\alpha_2 - 2 \alpha}{15 \alpha_1 - 15 \alpha_2} \, dx \right] \)

Since \( w_0(x) \) is a polynomial, we need \( \lim_{x \to \infty} \frac{\exp(2x - g f'' + 3x)}{x} = 0 \), there are obviously polynomials for \( g \) or powers such that \( w_0(x) \) doesn't vanish fast enough.

Also \( w_0(x) \) is never zero (since \( \exp(x) \to \infty \) and the roots are complex so only \( = 0 \) for values of \( x \) off the real line).

Thus complex roots won't work.

For real roots: Using \( C_1, C_2, C_3 \) we can make \( \alpha(x) = 1-x^2 \)

\[ C_1: L \Rightarrow \frac{4x}{4x^2+4x+4} L \quad C_2: x \to \frac{1}{2} \log \left( \frac{x^2-1}{x^2+1} \right) \quad C_3: x \to -\frac{1}{4} \int \frac{dx}{x^2+1} \]

\[ \alpha_0 = x^2 + \alpha, x + \alpha \Rightarrow \frac{\alpha_0}{\alpha_0} (w_0) (x + \alpha_0) \Rightarrow \frac{\alpha_0}{\alpha_0} \left( \int \frac{1}{4x^2+4x+4} \, dx \right) \]

\[ = -\left( \frac{x^2}{4x^2+4x+4} + \frac{1}{2} \log \left( \frac{x^2-1}{x^2+1} \right) \right) \]

\[ = -x^2 + 1 \]

We can also use \( \beta = q - \alpha, \Delta = -(p + q + \alpha) \) (just a redefinition) and then things clean up nicely:

\[ \frac{\beta}{\alpha} = \frac{q + 1}{1-x^2} \]

\[ w_0 = C \exp \left[ \int \frac{4x+4}{x^2+ax+4a} \, dx \right] \]

\[ = C \left( (1+x)^{g-1/1-x^{2}} \right) \]

\[ = C \left( (1-x)^{g/1-x^{2}} \right) \]

Now, once again \( w_0(x) \) won't go to zero fast enough to beat higher powers of \( x \) in \( f(x) \).

But the roots of \( \alpha \) at \( x = \pm 1 \) allow us to use the nonzero part between those and glue it to \( 0 \) for \( x \in [-1, 1] \) (the box). So we are led to \( f(-1, 1] \) though this can easily be shifted to any \( [a, b] \) by adjusting \( c_1, c_2, c_3 \).
\[ w(x) = (1+x)^p(1-x)^q \] works and gives rise to the Jacobi polynomials \( J_n^p(x) \) where \( p, q \) refer to the values in \( w(x) \). That is, you pick \( p \) and \( q \) and using \( w(x) \) find an entire set of polynomials specific to these \( p \) and \( q \).

Special cases:

- If \( p = q = m \) (\( m > 0 \)) with \( w(x) = (1-x^2)^m \) \( \Rightarrow \) Gegenbauer polynomials \( G_n^m(x) = J_n^m(x) \).

- If \( p = q = -\frac{1}{2} \) with \( w(x) = (1-x^2)^{-1/2} \) \( \Rightarrow \) Chebyshev polynomials \( T_n(x) = J_n(x) \).

- If \( p = q = 0 \) with \( w(x) = 1 \) \( \Rightarrow \) Legendre polynomials \( P_n(x) = J_n^0(x) \).

\( \alpha(x) \) is linear:

\( \alpha(x) = x + \alpha_2 (b \gamma C_1) \Rightarrow \alpha(x) = x \) \( (b \gamma C_1) \Rightarrow \alpha(x) = 0 \) \( \quad \text{at} \; x = 0 \)

Then:

\[ w(x) = C \exp \left[ -\frac{a}{x} \right] \] \( = C \exp \left[ -\frac{a}{x} \right] \] \( = C \cdot x^{-a} \cdot e^{-ax} \)

If \( a < 0 \) then \( w(x) \to 0 \) fast enough \( w(x) \to 0 \) for any power of \( x \) in \( f(x) \), but of course as \( x \to -\infty \) then \( w(x) \to 0 \).

So we use the root at \( x = 0 \) to pair it up with \( 0 \) for all \( x < 0 \) (Heaviside).

Using \( C_2 \) to set \( B_0 = -1 \) and calling \( B_3 = s+1 \) \( \text{w} \) \( s > 1 \) then \( w(x) = x^s e^{-x} \) for \( C_0 \).

For \( s = 0 \) these are the Legendre polynomials \( L_n(x) = L_n^0(x) \).

For \( s > 0 \) they are the associated Legendre polynomials \( L_n^s(x) \).
The Purfect Sturm

\( a(x) \) is constant:

\[ C_1 \Rightarrow a_0 = 1 \Rightarrow a(x) = 1 \Rightarrow \omega(x) = C \exp \left[ \int \left( a_0 x + b_1 \right) \, dx \right] \]

\[ = C \exp \left[ \frac{b_1}{2} x^2 + b_1 x \right] \]

\[ = \exp \left( -\frac{b_1}{2a_0} \right) C \exp \left[ \frac{b_1}{a_0} x \left( x + \frac{b_1}{a_0} \right) \right] \]

If \( b_1 < 0 \) then \( \omega = \omega \) falls exponentially as \( |x| \to \infty \), so it works!

With \( C_2 \) and \( C_3 \) we make \( b_0 = -2, b_1 = 0 \) and \( c' = 1 \) \( \Rightarrow \omega(x) = e^{x^2/2} \) \( |x| \to \infty, \infty \)

In this case \( Q_n(x) \) are the Hermite.