Okay, so let's go back to the finite dimensional version of this for a clearer picture.

Say the system being studied is described by a vector in a 5D, call it $\phi$. Now we can express the vector in terms of components along an orthonormal basis, but which one? Well it depends on what we want to measure about our system. Suppose we want to measure $\hat{A}$ which has a corresponding self-adjoint operator $\hat{A}$ in the 5D vector space.

Furthermore, suppose that $\hat{A}$ has 5 distinct eigenvalues $\{\lambda_i\}$, it must have 5 orthogonal eigenvectors $\{|\phi_i\rangle\}$ because its s.a. So we can express $\phi = \sum_{i=1}^{5} c_i |\phi_i\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle + c_3 |\phi_3\rangle + c_4 |\phi_4\rangle + c_5 |\phi_5\rangle$.

Now to obtain this, we could have used the resolution of the identity affiliated w/ $\hat{A}$, i.e. $I = \sum_{i=1}^{5} |\phi_i\rangle \langle \phi_i|$ or $I \phi = \phi_1(\phi_1, \phi) + \phi_2(\phi_2, \phi) + \phi_3(\phi_3, \phi) + \phi_4(\phi_4, \phi) + \phi_5(\phi_5, \phi) = \phi_1 c_1 + \phi_2 c_2 + \phi_3 c_3 + \phi_4 c_4 + \phi_5 c_5$.

Now realize that $I$ is just a set of projection operators $P_i \phi = \phi_i(\phi_i, \phi)$ which give the component of $\phi$ along $|\phi_i\rangle$, i.e. $\phi_1 c_1 + \phi_2 c_2 + \phi_3 c_3 + \phi_4 c_4 + \phi_5 c_5 = \sum_{i=1}^{5} c_i \phi_i$.

Now is there anything important about the labelling 1, 2, 3, 4, 5? No, but we can label it in terms of increasing eigenvalues, i.e. $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$.

Having done so we can now build $E(\lambda) = \left\{ \begin{array}{ll} 0 & \lambda < \lambda_1 \\ \sum_{i=1}^{2} P_i & \lambda_1 \leq \lambda \leq \lambda_2 \\ \sum_{i=1}^{2} P_i & \lambda_2 \leq \lambda \leq \lambda_3 \\ \sum_{i=1}^{2} P_i & \lambda_3 \leq \lambda \leq \lambda_4 \\ \sum_{i=1}^{2} P_i & \lambda_4 \leq \lambda \leq \lambda_5 \end{array} \right.$ where $\lambda$ is a continuous variable from $-\infty$ to $\infty$.

Notice that $E(\lambda)$ is intimately affiliated w/ $\hat{A}$! Turning the $\lambda$-axle, $E(-\infty) = 0$, $E(\infty) = I$.

Now to interpret $A_{\lambda}$ we need one more piece of the story: First of all $\langle \phi, \phi \rangle = \sum_{i=1}^{5} |c_i|^2 = 1$ which means that we can interpret $|c_i|^2$ as the probability that the vector would be realized in state $|\phi_i\rangle$ upon measurement.

In other words $|c_i|^2$ is the probability that measuring $\hat{A}$ on $\phi$ would give $\lambda_i$, that is $|c_i|^2$ is the probability of getting $\lambda_i$.

What about the probability of getting $\lambda_6$ or $\lambda_7$? Obviously, its $0$, but we can write this as $||[E(\lambda_6) - E(\lambda_5)] \phi||^2 = ||[\hat{P}_1 + \hat{P}_2 + \hat{P}_3 - \hat{P}_4] \phi||^2 = ||c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3||^2 = (c_1^2 + c_2^2 + c_3^2) + (c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2) = |c_1|^2 + |c_2|^2 + |c_3|^2$. 

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Moving along:

\[ \langle A \rangle = \int_\omega \sum \lambda_i \langle \psi | [E(\lambda_i + \alpha) - E(\lambda_i)] \phi | \psi \rangle \]

That is, the average outcome of many measurements of \( A \) on a large collection of systems in \( \omega \).

Each \( \lambda_i \) is weighted by its probability.

Going back to our simple model, the only contributions will come from \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) so

\[ \langle A \rangle = \lambda_1 |c_1|^2 + \lambda_2 |c_2|^2 + \lambda_3 |c_3|^2 + \lambda_4 |c_4|^2 + \lambda_5 |c_5|^2 \]

But:

\[ \langle A \rangle = \langle \psi | A | \psi \rangle \]

Proof:

\[ \langle A \rangle = \langle \psi | \sum \lambda_i dE(\lambda) | \psi \rangle \]

\[ = \int_\omega \lambda d [ \langle \psi | E(\lambda) | \psi \rangle ] \]

\[ = \int_\omega \lambda d \sum \lambda_i \langle \psi | [E(\lambda_i + \alpha) - E(\lambda_i)] \phi | \psi \rangle \]

But since \( E(\lambda) \) is composed of projections, \([E(\lambda_i + \alpha) - E(\lambda_i)] = [E(\lambda_i + \alpha) - E(\lambda_i)]^2\]

and since the \( E(\lambda_i)'s \) are self-adjoint:

\[ \langle A \rangle = \int_\omega \sum \lambda_i \langle \psi | [E(\lambda_i + \alpha) - E(\lambda_i)] \phi | \psi \rangle \]

\[ = \int_\omega \sum \lambda_i \langle \psi | [E(\lambda_i + \alpha) - E(\lambda_i)] \phi | \psi \rangle \]

\[ = \langle A \rangle \]

Once again:

\[ \langle A \rangle = \sum \lambda_i |c_i|^2 \]

\[ = \sum \lambda_i |c_i|^2 = \langle A \rangle \]
Furthermore we can define the mean square deviation which measures deviation from the mean \( \langle A \rangle \):
\[
(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle
\]
which leads to:
\[
(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2
\]
Proof: \[
(\Delta A)^2 = \langle [A - \langle A \rangle]^2 \rangle = \langle [\phi, [A - \langle A \rangle]\phi] \rangle
\]
\[
= \langle \phi, [A^2 - A\langle A \rangle + \langle A \rangle^2] \phi \rangle
\]
\[
= \langle \phi, A^2 \phi \rangle - 2 \langle \phi, A\phi \rangle \langle A \rangle + \langle A \rangle^2
\]
\[
= \langle A^2 \rangle - \langle A \rangle^2
\]
Now \[
(\Delta A)^2 = \langle [A - \langle A \rangle]^2 \rangle = \langle [A - \langle A \rangle]\phi, [A - \langle A \rangle]\phi \rangle
\]
\[
= \| [A - \langle A \rangle]\phi \|^2
\]
But this means \( \Delta A = 0 \) \( \iff \) \( [A - \langle A \rangle]\phi = 0 \) \( \iff \) \( A\phi = \langle A \rangle\phi \) so \( \Delta A = 0 \) only for \( \phi \) to be an eigenstate.

We can generalize to multiple measurements at different times.
\[
A \equiv \text{ Let } A, B, C \text{ be observables s.t. } [A, B] = [A, C] = [B, C] = 0.
\]
Then \[
P(a_1, a_2 \mid b_1, b_2 ; c_1, c_2) = \| [E_a(b_1) - E_a(b_2)][E_b(c_1) - E_b(c_2)][E_c(a_1) - E_c(a_2)] \phi \|^2
\]
where \( E_Q(q) \) are resolutions of \( I \) w.r.t. \( J \), order doesn't matter due to

Everything so far is at an instant of time. The question then is “how does the story evolve w time?”

For every system there exists a Hermitian operator \( H \) (the Hamiltonian) from which:
\[
H \Psi(x, t) = \frac{\partial \Psi(x, t)}{\partial t}
\]
provided no measurements are made!

Now this leads to:
\[
\frac{\partial}{\partial t} \langle \Psi, \Psi \rangle = 0
\]
Proof: \[
\frac{\partial}{\partial t} \langle \Psi, \Psi \rangle = \frac{\partial}{\partial t} \left( \frac{\Psi}{\nabla^2} \right) = \left( \frac{1}{i\hbar} \nabla \Psi, \nabla \Psi \right) + \left( \frac{1}{i\hbar} \nabla \Psi, \nabla \Psi \right) = -\frac{i}{\hbar} \left( \nabla^2 \Psi, \nabla \Psi \right) + \frac{i}{\hbar} \left( \nabla \Psi, \nabla \Psi \right) = 0
\]
This implies that \( A \nabla \) is commutable with \( A \nabla \).

Furthermore, if \( H \) does not depend on \( t \), then \( \tilde{\Psi}(x, t) = e^{-iHt} \tilde{\Psi}(x, 0) \), from which one can show that if \( [A, H] = 0 \) w/ H time ind., then measurement results of \( A \) will be time ind. as well.
And we finish with a means of gaining all the information needed to determine \( \Psi \).

\[ \mathbf{A} \mathbf{V} \]

If we choose one measure \( A, B, C \) and find we indeed certainty \( \alpha E_a, \beta E_b, \gamma E_c, \delta E_d \), then

\[ [E_a(\Phi_a) - E_a(\phi_a)] [E_b(\Phi_b) - E_b(\phi_b)] [E_c(\Phi_c) - E_c(\phi_c)] \Psi = \Psi \]

That is, the measurements project the original wavefunction onto a subspace of Hilbert

associated with the projector \([E_a(\Phi_a) - E_a(\phi_a)] [E_b(\Phi_b) - E_b(\phi_b)] [E_c(\Phi_c) - E_c(\phi_c)]\).

Start with a single measurement of \( A \). If there is only one value between \( a_1 \) and \( a_2 \), then \( \Psi = e^{i\alpha} \Phi_a \), (\( \alpha \) is non-deg) where \( \Phi_a \) is a normalized eigenvector of \( A \) belonging to \( a \), and \( e^{i\alpha} \) is an indeterminate phase factor.

Now, if \( a \) is degenerate, then the measurement has projected us into a subspace of Hilbert space which is spanned by the orthonormal eigenvectors associated with \( a \), i.e., \( \Psi = \sum \phi^m \Phi^m \) where \( n > 1 \) is the degeneracy of \( a \). So \( \Psi \) is not well defined. But we can solve this by measuring \( B \) where \( [A, B] = 0 \).

If this doesn't resolve it, then measure \( C \) where \( [A, C] = 0 \) and so on.

Returning to our simple example: If \( A \phi = \lambda \phi, \beta \phi_1, \gamma \phi_2 + \delta \phi_3 + \beta \phi_4 \), then measuring \( A \) is not enough since \( \lambda \) could mean \( \phi, \phi_1, \phi_2, \phi_3, \phi_4 \), and \( \beta \) could mean \( \phi, \phi_1, \phi_2, \phi_3, \phi_4 \). But imagine \( B \) where \( [A, B] = 0 \). If \( B \phi = \gamma \phi_1, \delta \phi_2 + \beta \phi_3 + \gamma \phi_4 + \beta \phi_5 \), then knowing the eigenvalues of \( A \) and \( B \) (or measuring both) is enough to specify the state.

In the end, to fully specify \( \Psi \), we need the results of a complete set of commuting observables.

Example: Electron in hydrogen can have \( H, L, S, J, \) \( J_z \) to fully specify the state. Alternatively, \( H, L, S, J, J_z \) \( J = L + S \) works.