As a follow-up to Fourier series:

Clearly, we can adjust the interval from \([-\pi, \pi]\) to \([-L, L]\) with:

\[
f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) dx
\]

And we can also shift the interval from \([-\pi, \pi]\) to \([-\pi + d, \pi + d]\) with:

\[
f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi (x-d)}{L}\right) dx
\]

For even functions \(a_n = 0\) \(\cos(n\pi x)\) series

For odd functions \(a_n = 0\) \(\sin(n\pi x)\) series

And lastly, the differential (or eigenvalue) equation satisfied by these is:

\[\frac{d^2 u}{dx^2} = -\alpha^2 u\]

More Fourier Fun... Opening our Minds Beyond Periodicity or Intervals

We have so far found that the Fourier series uniformly converges for continuous functions with piecewise continuous derivatives either over a finite interval \([a, b]\) with \(f(a) = f(b)\)

or everywhere for periodic functions \(f(x + L) = f(x)\).

But then there are certainly many functions that are not periodic that we would like to approximate over the whole real line \(x \in (-\infty, \infty)\).

Okay, so let's just try taking \(L \to \infty\). To get ready define:

\[
\left(\frac{\pi}{L}\right)^2 x \equiv \gamma, \quad \frac{\pi}{L} \gamma \equiv k_n
\]

Then our \([-L, L]\) set becomes:

\[
f(x) = \sum_{k=-\infty}^{\infty} g_k \cos(kx) dx
\]

\[
\gamma \equiv \frac{\pi}{L} x, \quad k \equiv \frac{\pi}{L} \gamma
\]

And now we take the limit as \(L \to \infty\), in which case the discrete steps \(\Delta k_n \to 0\). In this case \(\sum_{\infty} g_k = \int g(x) dx\).

Recall that in proving completeness:

\[
f^\prime(x) = \int g(x) e^{-i\gamma x} dx
\]

in Sturm-Liouville: we used that:

\[
g_{\infty}(x) = \int_{-\infty}^{\infty} f(\gamma) e^{-i\gamma x} d\gamma = 0
\]

\[
g_{\infty}(x) = \int_{-\infty}^{\infty} f(\gamma) e^{-i\gamma x} d\gamma
\]

\[
f(x) = \sum_{\infty} g_k(x) = \sum_{\infty} \frac{1}{i\pi} \int_{-\infty}^{\infty} f(\gamma) e^{-i\gamma x} d\gamma
\]
Finding \( g(\kappa) \) from \( f(\gamma) \):

\[ f(\gamma) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\gamma') e^{-ik' \gamma'} d\gamma' \right] e^{ik\gamma} dk \]

and then switching \( d\gamma' \leftrightarrow dk \)

\[ f(\gamma) = \int_{-\infty}^{\infty} f(\gamma') \left[ \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-ik' \gamma'} d\gamma' \right] d\gamma' \]

\[ \Rightarrow \delta(\gamma - \gamma') \]

One more time:

\[ \int_{-\infty}^{\infty} f(\gamma) f(\gamma') d\gamma = \int_{-\infty}^{\infty} \left[ \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa) e^{ik' \gamma} dk \right] f(\gamma') e^{-ik' \gamma} d\gamma' \]

Switching \( d\kappa' \leftrightarrow d\gamma \)

\[ = \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa) f(\kappa') \left[ \frac{1}{i2\pi} \int_{-\infty}^{\infty} e^{ik' \gamma} d\gamma \right] d\kappa' \]

\[ = \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa) f(\kappa') \delta(\kappa' - \kappa) d\kappa' \]

\[ = \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa) f(\kappa) d\kappa = \int_{-\infty}^{\infty} \delta(\kappa) f(\kappa) d\kappa \]

An interesting question to ask is: "Is the Fourier transform of a product of functions equal to the product of the Fourier transforms of the individual functions?" The answer, of course, is no, in part because the transforms involve calculus, whereas we know \( \frac{d}{dx}(uv) \neq \frac{dv}{dx}u + \frac{du}{dx}v \) \( u \neq 0 \). But what does it give?

\[ G(k) = \frac{1}{i2\pi} \int_{-\infty}^{\infty} f(\gamma) e^{-ik\gamma} d\gamma \]

\[ \frac{1}{i2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa') e^{ik' \gamma} d\kappa' \right] e^{-ik\gamma} d\gamma' \]

Switching \( d\kappa' \leftrightarrow d\gamma \)

\[ = \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa) f(\kappa') \left[ \frac{1}{i2\pi} \int_{-\infty}^{\infty} e^{ik' \gamma} d\gamma \right] d\kappa' \]

\[ = \frac{1}{i2\pi} \int_{-\infty}^{\infty} g(\kappa) f(\kappa') \delta(\kappa' - \kappa) d\kappa' \] which is called the "convolution" of \( g_1 \) and \( g_2 \).

In d-dimensions we have:

\[ F(\vec{r}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} G(\vec{k}) e^{-i\vec{k} \cdot \vec{r}} d\vec{k} \]

\[ G(\vec{k}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} F(\vec{r}) e^{i\vec{k} \cdot \vec{r}} d\vec{r} \]

and of course

\[ \delta^d(\vec{r} - \vec{r}_0) = \delta(x_1 - x_{10}) \delta(x_2 - x_{20}) \cdots \delta(x_d - x_{d0}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\vec{k} \cdot (\vec{r} - \vec{r}_0)} d\vec{k} \]
Okay let's review our process.

First we use Weierstrass to argue that if \( f(x) \) is continuous on \([a, b]\) there exists a sequence of polynomials \( P_n(x) \) such that \( \lim_{n \to \infty} P_n(x) = f(x) \) with uniform (hence also mean) convergence.

Then we take the linear set of which \( P_n(x) \) is a superposition and G.S. if \( f \) needed to get an orthogonal basis \( \{ Q_n \} \). Then \( P_n(x) = \sum_{i=0}^{n} c_{i,n} Q_i(x) \). The \( c_{i,n} \) depend on \( n \).

We then prove the completeness of this basis via closure using \( (f, Q_n) = 0 \iff f = 0 \), but \( (f, Q_n) = 0 \implies (f, P_n) = 0 \) but we know that \( P_n \) converges in the mean to \( f \) so \( \| f - P_n \| < \varepsilon \)

hence \( \| f(x) + P_n(x) \| < \varepsilon \iff f = 0 \) almost everywhere.

But since \( \{ Q_n \} \) is complete and orthonormal, then \( f(x) = \sum_{i=0}^{\infty} c_i Q_i(x) \) which converges in the mean \( f(x) = \sum_{i=0}^{\infty} c_i Q_i(x) \).

This is the story for finite intervals, but can be generalized via SL to non-general settings.
Let's repeat the Fourier trickery of going from \( x, y \to \cos(\theta), \sin(\theta) \), but this time in \( 2D \).

We have \( F(\phi) = \int_{\mathbb{R}^2} f(x, y) e^{i\phi x^2 + i\beta y^2} dx dy \) with uniform (mean) convergence.

Upon redefining \( 2: \) is a \( \sin \) combo of \( x, y \):

\[
\begin{align*}
&z \equiv x_1 + i x_2 = r \sin \theta e^{i \phi} \\
&z \equiv x_1 - i x_2 = r \sin \theta e^{-i \phi} \\
&x_1 = r \cos \theta \\
&\theta \in [0, \pi]
\end{align*}
\]

Down a rabbit hole:

Now just like we Fourier we want to restrict to \( r = 1 \) and relabel \( x, y \to \theta \). We know that

\[
\alpha, \beta \geq 0 \implies \alpha + \beta \geq 0 \quad \text{and} \quad \alpha + \beta \leq 1 \implies \alpha \leq 1 - \alpha \leq \beta \leq 1 - \beta.
\]

Now if \( \alpha > 0 \) then \( \alpha + \beta - 1 \leq 0 \) and \( \alpha + \beta - 1 > 0 \) if \( \beta > 0 \) again.

We can now rewrite \( \sin \theta \cos \theta = \frac{1}{2} (\sin(2\theta) - \sin(0)) \)

\[
\begin{align*}
&= (1 - \cos \theta) \frac{1}{2} (\sin(2\theta) - \sin(0)) \\
&= (1 - \cos \theta) \frac{1}{2} \sin(2\theta)
\end{align*}
\]

Now if we relabel \( \omega \) as our label we need summation limits. Using \( \alpha, \beta > 0 \):

First: \( m = \alpha - \beta \leq l \) \( \implies \alpha + \beta - 1 \leq 0 \)

\[
\alpha \leq 1 - \alpha + \beta \\
\beta \leq 1 - \alpha
\]

Therefore \( \alpha \leq 1 \) and we have:

\[
F_m(\phi) = \sum_{\alpha=1}^{\infty} \sum_{\beta=0}^{\alpha-2} B_{\alpha-2}(\phi) \sin \theta = \sum_{\alpha=1}^{\infty} \sum_{\beta=0}^{\alpha-2} B_{\alpha-2}(\phi) \sin \theta
\]

Lecture20- More Fourier and his Marriage to Legendre Page 4
Now we would like to C.S. the $Y_{nm} (\theta, \phi)$ . That is we want $Y_{nm} (\theta, \phi)$ s.t. 

$$\langle Y_{nm}, Y_{m} \rangle = \int_{\mathbb{S}^2} Y_{nm}^* Y_{m} \, d\Omega = \delta_{nm}$$

Starting w/ $Y_{00}$ we first recall that $f_{lm} (\cos \theta)$ has degree $l$-1 and thus is constant for $l=0$ = 1. Then $Y_{00} = C \int_{0}^{\pi} \int_{0}^{2\pi} C \cos \theta \, d\theta d\phi = \int_{0}^{\pi} \cos \theta \, d\theta = 0 \Rightarrow Y_{00} (\theta, \phi) = \frac{1}{\sqrt{\pi}}$

Now for $l=1$ we have 

$m = 0 \quad f_{10} (\cos \theta) = A + iB \cos \theta \Rightarrow Y_{10} (A + iB \cos \theta)

m = -1 \quad f_{1-1} (\cos \theta) = \text{constant} - C \Rightarrow Y_{1-1} = e^{-i\phi}

m = 1 \quad f_{11} (\cos \theta) = \text{constant} + i \Rightarrow Y_{11} = e^{i\phi}$

Let us do $Y_{10}$, $Y_{10} = \frac{Y_{00} - Y_{01}}{\|Y_{00} - Y_{01}\|}_{\mathbb{S}^2} = \left[ A + iB \cos \theta - \frac{1}{\sqrt{\pi}} \int_{0}^{\pi} \int_{0}^{2\pi} \cos (A + iB \cos \theta) d\theta d\phi \right]_{\|}$

$$= \left[ A + iB \cos \theta - \frac{2\pi}{\sqrt{\pi}} (A + O) \right]_{\|}$$

$$= \frac{B \cos \theta}{\sqrt{\int_{0}^{\pi} \int_{0}^{2\pi} \cos^2 \theta d\theta d\phi}}$$

$$= \frac{B \cos \theta}{\sqrt{3\pi}}$$

Similarly $Y_{1-1} = \frac{A}{\sqrt{\pi}} e^{-i\phi}$, $Y_{11} = \frac{A}{\sqrt{\pi}} e^{i\phi}$

The $Y_{nm} (\theta, \phi)$ are the spherical harmonics. These are obviously orthogonal. To prove they are complete we just recall from Wigner's that the $Y_{nm} (\theta, \phi)$ provide a lin. ind. basis s.t. the sequence $F_{nm} (\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \beta_{lm} Y_{lm} (\theta, \phi)$ converges uniformly (and nearly in $F(\vec{r})$). Then the orthogonal set $Y_{nm} (\theta, \phi)$ is just a linear combination of the $Y_{nm} (\theta, \phi)$ so too are $Y_{nm}$ lin. comb. of $Y_{nm}$.

For $Y_{nm}$ we know that $|F(\vec{r})| - |F_{nm} (\vec{r})| < 0$ and if $( F_{nm} (\vec{r}), F_{nm} (\vec{r}) ) = 0$

then $\| F(\vec{r}) \| + \| F_{nm} (\vec{r}) \| < \| F(\vec{r}) \| = 0$ almost everywhere $= Y_{nm}$ and hence $Y_{nm}$ are complete w/ regards to functions of $x, y, z$ s.t. $x^2 + y^2 + z^2 = 1$. Thus

$$F(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \beta_{lm} Y_{lm} (\theta, \phi)$$

does not depend on each other.
A bit of “geometry”. The unit sphere $S^d$ we can think of as a sequence of circles along a line that goes from $[-1, 1]$ and the circles begin and end at $r=0$ and swell to $r=1$ in the middle.

![Diagram of a sequence of circles along a line]

Then, in a certain sense, we could imagine the story that plays out here is a mix of Fourier functions for the circle d.o.f., and Legendre modes for the $[-1,1]$ d.o.f.

In fact, the “Rodrigue” formula in this case is:

$$Y_{k \mu}(\theta, \phi) = (-1)^{k + \mu} \left( \frac{2k + 1}{4\pi} \right) \frac{(k - \mu)!}{(k + \mu)!} \hat{P}_{k + \mu}(\cos \theta) \sin^{\mu} \theta \quad \text{for } \mu \geq 0 \quad \text{and} \quad Y_{k \mu} = (-1)^{k + \mu} Y_{-k - \mu}$$

where

$$\hat{P}_{k + \mu}(x) = (1-x^2)^{\frac{\mu}{2}} \frac{d^\mu}{dx^\mu} P_{k}(x)$$

where $P_k(x)$ are the Legendre polynomials, and of course the $\hat{P}_k$ are the Fourier functions.

In fact, removing the Fourier part by setting $\mu = 0$ gives $Y_{k0}(\theta, \phi) = \left( \frac{2k + 1}{4\pi} \right)^{\frac{1}{2}} P_k(\cos \theta)$

In fact the $\hat{P}_k$ are just the associated Legendre functions.