Greens functions are a means of transforming differential equations into integral equations.

Sometimes this is easy: \( \frac{d}{dx} = f(x) \) \( \gamma(0) = \gamma_0 \Rightarrow \gamma(x) = \gamma_0 + \int_0^x f(x') dx' \)

Sometimes this is not: \( \frac{d}{dx} = f(x') \) \( \gamma(0) = \gamma_0 \Rightarrow \gamma(x) = \gamma_0 + \int_0^x f(x') dx' \) \( \frac{d^2}{dx^2} = f(x) \Rightarrow \) ?

Starting out quite generally, consider \( Ly = f \) where \( L \) is a linear ordinary differential operator, i.e., \( a_0(x) y + a_1(x) \frac{dy}{dx} + a_2(x) \frac{d^2y}{dx^2} + \cdots + a_n(x) \frac{d^ny}{dx^n} = f(x) \) and \( f \) is a given function of \( x \).

Suppose that \( L \) possesses a complete and orthonormal set of eigenfunctions: \( L \phi_n(x) = \lambda_n \phi_n(x) \)

Because they are complete, we know: \( \gamma(x) = \sum_n \alpha_n \phi_n(x) \) and \( f(x) = \sum_n \beta_n \phi_n(x) \)

Showing these in:
\[
Ly = \sum_n \lambda_n \phi_n(x) = \sum_n \alpha_n \phi_n(x) = \sum \lambda_n \phi_n(x) = \sum_n \beta_n \phi_n(x) = 0
\]

(For since \( \phi_n \) are lin. ind., \( \lambda_n \phi_n \) \( \Delta_n \phi_n = 0 \Rightarrow \alpha_n = \sum \frac{\lambda_n}{\phi_n} \phi_n(x) \)

Now recall that \( A_n = (\phi_n, f) \) \( \Rightarrow \gamma(x) = \sum_n \frac{1}{\lambda_n} \phi_n(x) (\phi_n, f) = \int \sum_n \frac{\phi_n(x)^2}{\lambda_n} f(x') x' \)

\( G(x, x') = \) Greens Function

Considers: \( LG(x, x') = \sum_n \frac{L \phi_n(x_0) \phi_n(x_0)}{\lambda_n} = \sum_n \phi_n(x) \phi_n(x_0) \equiv L(x, x') \) for \( L \)

\( B_n: \) \( \int \phi_n(x') f(x') dx' = \sum_n \phi_n(x) \phi_n(x') f(x') dx' = \sum_n \phi_n(x) (\phi_n, f) = f(x) \)

\( \therefore \) \( \int \phi_n(x') f(x') dx' = \sum_n \phi_n(x) (\phi_n, f) = f(x) \)

\( \therefore \) \( L \gamma = f \) \( \Rightarrow \gamma = L^{-1} f = \int G(x, x') f(x') dx' \)

Another way this is written is: \( L \gamma = f \) \( \Rightarrow \) \( K = L^{-1} f = \int G(x, x') f(x') dx' \)

Note this is the identity, not \( I(x, x') \)

(Interpretation of \( f \))
Let us view this in a simple example:

\[ L = \frac{d}{dx} \Rightarrow L \gamma = f \quad \text{with} \quad \gamma(0) = \gamma_a, \quad \gamma(1) = \gamma_b \Rightarrow \gamma(x) = \gamma_a + \int_a^x \gamma(x') f(x') \, dx' \]

\[
\frac{d}{dx} \gamma = f \quad \text{with} \quad \gamma(0) = \gamma_a, \quad \gamma(1) = \gamma_b, \quad \gamma(x) = \int_a^b \theta(x-x') f(x') \, dx' \]

for \( x \in [a, b] \). Thus:

\[
\theta(x-x') = \begin{cases} 
1 & x < x' \\
0 & x > x'
\end{cases}
\]

Then:

\[
G(x,x') = \theta(x-x')
\]

and recall that:

\[
L G(x,x') = \delta(x-x') \Rightarrow \frac{d}{dx} \left( \int_0^b \theta(x-x') f(x') \, dx' \right) = \delta(x-x')
\]

Let's write another:

\[ L = \frac{d^2}{dx^2} \Rightarrow L \gamma = f \quad \text{with} \quad \gamma(0) = \gamma_a, \quad \gamma(1) = \gamma_b, \quad \gamma''(x) = \gamma_a + \int_0^x f(x') \, dx'
\]

\[
\gamma(x) = \gamma_a + \int_0^x f(x') \, dx' + \int_0^x \left( \gamma_a \theta(x-x') + \int_0^x \theta(x-x') f(x') \, dx' \right) \, dx'
\]

And since:

\[
L \delta(x-x') = \delta(x-x') \Rightarrow \frac{d}{dx} \left( \int_0^b \theta(x-x') f(x') \, dx' \right) = \delta(x-x')
\]

And once more:

\[ L = \frac{d^2}{dx^2} \Rightarrow L \gamma = f \quad \text{with} \quad \gamma(0) = \gamma_a, \quad \gamma(1) = \gamma_b, \quad \gamma''(x) = \gamma_a + \int_0^x \theta(x-x') f(x') \, dx'
\]

\[
G(x,x') = -x(1-x') + (x-x') \theta(x-x') = \begin{cases} 
-x'(1-x) & 0 \leq x' \leq x \quad (\theta = 1) \\
-x(1-x') & x < x' \leq 1 \quad (\theta = 0)
\end{cases}
\]

And once again:

\[ L G(x,x') = \delta(x-x') \]
To be more general, consider: \( \frac{d^2}{dx^2} + a \frac{d}{dx} + b y(x) = f(x) \) \( y(x) \in (-\infty, \infty) \)

Now let's assume that \( f(x) \) and \( y(x) \) all \( \to 0 \) for \( |x| \to \infty \). Therefore we can do Fourier transforms of each:

\[
\int_{-\infty}^{\infty} \left( \frac{d^2}{dx^2} + a \frac{d}{dx} + b y(x) \right) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx
\]

Then we have:

\[
\int_{-\infty}^{\infty} \left( a - i \frac{d}{dx} \right) \frac{d}{dx} e^{ikx} + \frac{1}{k} \gamma(0) e^{ikx} dx = \frac{1}{k} \int_{-\infty}^{\infty} f(x) e^{ikx} dx
\]

So in the end we have:

\[
\left( -a + i \frac{d}{dx} \right) \gamma(0) = \frac{1}{k} \int_{-\infty}^{\infty} f(x) e^{ikx} dx
\]

\[
\gamma(0) = \frac{1}{k} \int_{-\infty}^{\infty} f(x) e^{ikx} dx
\]

Therefore:

\[
G(x, x') = \int_{-\infty}^{\infty} e^{-ik(x-x')} \frac{dk}{2\pi} \quad y(x) = \frac{1}{k^2 + i\lambda k + 1} e^{ikx} \]

To do the integral we resort to the result of the last chapter. Consider the contour:

\[
\Gamma = -\frac{1}{2\pi} \int_{R} \frac{e^{-ik(x-x')}}{(k^2 + i\lambda k + 1)} dk
\]

\[
= \frac{1}{2\pi} \int_{-R}^{R} e^{-ik(x-x')} \frac{dk}{k^2 + i\lambda k + 1} \quad \text{as \( R \to \infty \)}
\]

First of all: \( \frac{e^{-ik(x-x')}}{(k^2 + i\lambda k + 1)} dk = 2\pi i \left( R_+ + R_- \right) \) where \( R_\pm \) is the residue coming from \( k_\pm \),

\[
R_+ = \frac{1}{2\pi} \int_{-\infty}^{-1} \frac{e^{-ik(x-x')}}{(k^2 + i\lambda k + 1)} dk \quad \text{where \( C_2 \) includes \( k_+ \) but not \( k_- \) (since the numerator is analytic)}
\]
Using \( \text{CIF} : \omega(z_0) = \frac{1}{2\pi i} \oint_{C_0} \frac{\omega(z)}{z - z_0} \, dz \)

Then

\[
R_+ = \frac{e^{-ik_+ (x-x')}}{k_+ - k_-} = \frac{e^{-i\left(\frac{1}{2}\sqrt{b-a^2} - i\frac{a}{2}\right)(x-x')}}{\sqrt{b-a^2}}
\]

\[
W_+ = \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-a^2}} \sin \left[ \sqrt{b-a^2} (x-x') \right]
\]

We need:

\[
R_+ + R_- = \frac{2i}{\sqrt{b-a^2}} \left[ -\cos \left( \frac{a}{2} \right) \sin \left( \frac{a}{2} \right) + \cos \left( \frac{a}{2} \right) \right]
\]

\[
= \frac{i e^{\frac{a}{2}(x-x')}}{\sqrt{b-a^2}} \sin \left[ \sqrt{b-a^2} (x-x') \right]
\]

Thus:

\[
I = \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-a^2}} \sin \left[ \sqrt{b-a^2} (x-x') \right]
\]

which also equals:

\[
I = \mathcal{G}(x,x') - \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_R} \frac{e^{-ik_+ (x-x')} e^{ik_- (x-x')}}{(R e^{i\theta})^2} \, d\theta
\]

\[
= 0 \text{ by Jordan's Lemma, but only for } x-x' > 0
\]

So we finally have:

\[
\mathcal{G}(x,x') = \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-a^2}} \sin \left[ \sqrt{b-a^2} (x-x') \right] \text{ for } x > x'
\]

On the other hand, for \( x < x' \) we alternatively use:

For which \( I = 0 \Rightarrow \mathcal{G}(x,x') = 0 \).

Putting these together:

\[
\mathcal{G}(x,x') = \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-a^2}} \sin \left[ \sqrt{b-a^2} (x-x') \right]
\]

Now of course we can show that \( L \mathcal{G}(x,x') = \delta(x-x') \), which you will do.

To get the most general solution, let's add solutions of the homogeneous \( L \gamma(x) = 0 \), i.e.,

\[
\gamma_1(x) = e^{-\frac{a}{2}x} \sin \left[ \sqrt{b-a^2} x \right]
\]

\[
\gamma_2(x) = e^{-\frac{a}{2}x} \cos \left[ \sqrt{b-a^2} x \right]
\]

So in total we have, as the general solution to \( L \gamma(x) = \frac{\partial^2 \gamma}{\partial x^2} + a \frac{\partial \gamma}{\partial x} + b \gamma = f(x) \) with \( f(x) > 0 \) everywhere,

\[
\gamma(x) = A \gamma_1(x) + B \gamma_2(x) + \left\{ \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-a^2}} \sin \left[ \sqrt{b-a^2} (x-x') \right] \right\} f(x') \, dx'
\]