Now what we just said is how to determine if and when a matrix has an inverse.
Recall that linear transformations can always be isomorphically mapped to matrices.
So the conditions for an inverse then are being reflected in the matrix conditions.

Consider a rotation acting on $\mathbb{R}^d$ which can be represented by

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Clearly $\det R_\theta = \cos^2 \theta + \sin^2 \theta = 1 \neq 0$ and so $R_\theta^{-1} = \frac{\text{adj} R_\theta}{\det R_\theta} = \frac{1}{\cos \theta \sin \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^T$

$= \begin{pmatrix} \frac{1}{\cos \theta} & \frac{1}{\sin \theta} \\ -\frac{1}{\sin \theta} & \frac{1}{\cos \theta} \end{pmatrix}$

so we need $T$ to get $\text{adj}(R_\theta)$.

But now consider the operator $D$ on $\mathbb{R}^d$ which we determined had no inverse since it wasn’t 1-to-1 on $\mathbb{R}^d$. What about the matrix reflection?

Well recall that for $P$, $D$ takes the form $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the $e_1, e_2$ basis,
and $D = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ in the $\{e_1 \pm e_2\}$ basis. From this, without taking the determinant formally we find:

From property 2 - in the $e_1, e_2$ basis clearly $\det D = 0 \Rightarrow D^{-1}$ does not exist.

From property 5 - the two rows of the $\{e_1 \pm e_2\}$ are the same, hence $\det D = 0 \Rightarrow D^{-1}$

From property 8 - since the two rows of the $\{e_1 \pm e_2\}$ are linearly dependent $\det D = 0 \Rightarrow D^{-1}$

From not being lazy - taking the determinant in either basis $\det D = 0 \Rightarrow D^{-1}$

On the whole, the ability to find the inverse of a linear transformation helps when trying to solve equations of the form:

$Ax = y$ where $A$ and $y$ are known and $x$ is unknown. Then $x = A^{-1} y$. 
Now a linear transformation $\mathbf{A}$ has acted on any vector $\mathbf{x} \in \mathbf{V}$ and returned another $\mathbf{y} \in \mathbf{V}$, and its action can be expressed in terms of a set of fixed basis vectors which do not undergo the action of the transformation. So the matrix form acts only on components, i.e. the $\alpha$-tuple. Now suppose we want to keep all vectors fixed, and just change the basis.

Then in we are led to the weirdness of “active” vs. “passive” transformations.

Note that “active” is how we define and described arbitrary linear transformations. However “passive” is obviously related to basis/coordinate changes. In some cases these can be exchanged (rotations) but in many cases not ($\mathbf{A} \neq \mathbf{0}$).

For arbitrary linear transformations $\mathbf{A}$, we can have $\mathbf{y} = \mathbf{A} \mathbf{x}$, whereas if all we are doing is a basis/coordinate transformation, then $\mathbf{x}' = \mathbf{B}^{-1} \mathbf{x} = \mathbf{x}$.

Now this might lead you to conclude that $\mathbf{B}^{-1} = \mathbf{I}$ (which is not far off!), but that is trying to interpret $\mathbf{B}^{-1}$ as an active linear transformation.

So here we go: $\mathbf{x} = \sum \alpha_i \mathbf{x}_i = \sum \mathbf{B}^{-1} \alpha_i \mathbf{B} \mathbf{x}_i$.

More technically: If $\mathbf{B}^{-1} = (b_{ij})$.

Now this means that we can only work with $\mathbf{B}$ for which $\mathbf{B}^{-1}$ exists!!

Going back to the active story, we may now take $\mathbf{y} = \mathbf{A} \mathbf{x}$ which all happens in $\mathbf{x}_i$, and translate it into $\mathbf{x}_i$:

Specifically: $\mathbf{y} = \sum \mathbf{B} \alpha_i \mathbf{x}_i = \sum \mathbf{B} \alpha_i$.

becomes $\mathbf{y}' = \sum \mathbf{B} \alpha_i \mathbf{x}_i = \sum \mathbf{B} \alpha_i \mathbf{B}^{-1} \mathbf{B} \mathbf{x}_i = \sum \mathbf{B} \alpha_i$.

or rewritten $\mathbf{B} \mathbf{B}^{-1} = \mathbf{I}$.

So in the end we have: $\mathbf{y} = \mathbf{A} \mathbf{x}$ acts as $\mathbf{B}^{-1} \mathbf{A} \mathbf{B}$.
The standard language is that $A'$ is "similar" to $A$. As a consequence, $\det A = \det A'$ and $[T_A] = [T_A']$.

Let's see this play out in an example.

Recall $D = \frac{d}{dt}$ on $P_1$ with $x = \alpha e + \alpha t = \frac{1}{2}(\alpha_0 + \alpha_1)(1+t) + \frac{1}{2}(\alpha_0 - \alpha_1)(1-t)$.

In the $(1, t)$ basis $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\alpha = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}$ so $\gamma = \lambda_1$

In the $(1 + t, 1 - t)$ basis $D' = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $\alpha' = \begin{pmatrix} \frac{1}{2}(\alpha_0 + \alpha_1) \\ \frac{1}{2}(\alpha_0 - \alpha_1) \end{pmatrix}$ so $\gamma' = \alpha_1$.

So what is the matrix that changes the basis $(1, t)$ to $(1 + t, 1 - t)$, or rather, the matrix which reflects how this acts on components?

It's just $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ so $B\alpha = \alpha'$ and $B^{-1}DB = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = D'$.

Recall that since $B$ acts on the basis, $B^{-1}$ acts on the components.

So in terms of the matrix action on the components, only we have:

$D\alpha = B B^{-1}DBB^{-1}\alpha = B B^{-1}DB\alpha' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix}.$
Okay, so normally a linear transformation (or isomorphic matrix) takes a vector to a different vector (or isomorphic n-tuple), but sometimes it gives back the same vector (in direction) albeit with a scalar multiple (which doesn't impact direction). The vector is called an eigenvector and the scalar an eigenvalue of the transformation.

Now let's collect what we are interested in. We consider a single linear transformation and then how it acts over an entire vector space. We are looking for special cases of this action.

We have: \[ A \mathbf{x} = \lambda \mathbf{x} \] (Note \( \mathbf{x} \), is not necessarily a basis vector in this case, but if it satisfies this equation it is an eigenvector).

Now this is an expression involving a linear transformation of vectors, but recall we map \( \mathbb{V} \to \mathbb{R}^n \), then \( A \to \text{matrix} \). So henceforth we will work with matrices acting on components in a given basis.

Let's consider solving problems for a moment.

1. \( A \mathbf{x} = \mathbf{y} \), where \( A \) and \( \mathbf{y} \) are known \( \Rightarrow \) requires us to figure out \( A^{-1} \) then \( \mathbf{x} = A^{-1} \mathbf{y} \)
2. \( A \mathbf{x} = \lambda \mathbf{x} \) has both \( \mathbf{x} \) and \( \lambda \) as unknowns. We need to work harder.

Consider: \( A \mathbf{x} = \lambda \mathbf{x} \Rightarrow (A - \lambda I) \mathbf{x} = 0 \); but wait if \( A \) is a matrix and \( \lambda \) a scalar we need: \( (A - \lambda I) \mathbf{x} = 0 \Rightarrow M \mathbf{x} = 0; \quad M = A - \lambda I \);

No imagine if \( M^{-1} \) existed, then \( \mathbf{x} = M^{-1} 0 = 0 \) and we're done.

To have an interesting \( (\neq 0) \) solution we need for \( M^{-1} \) not to exist.

The most robust predictor of whether \( M^{-1} \) exists is if \( \det A \neq 0 \) or not.

So, for \( \mathbf{x} \) to exist \( (\neq 0) \) we need \( \det (A - \lambda I) = 0 \); which only has \( \lambda \); unknowns.

Of course this is a scalar equation, but of order \( n \) (for an \( n \times n \) matrix) so we can expect up to \( n \) solutions (though not distinct), and we need to allow for complex ones since many (even totally real) polynomials need them, e.g., \( x^4 + 1 = 0 \Rightarrow x = \pm i \).

Once we have a solution for \( \lambda \), we plug it back into \( A \mathbf{x} = \lambda \mathbf{x} \) and solve for \( \mathbf{x} \).
An example:

Consider \( \mathbf{V} = \mathbb{R}^3 \) and the rotation transformation \( R_{xy} = \begin{pmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{pmatrix} \)

We can already determine one eigenvector, i.e. \( \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow R_{xy} \mathbf{x} = \mathbf{x} \) with eigenvalue \( \lambda = 1 \).

First of all, \( \det R_{xy} = 0 \) but that's okay since we need \( \det (R_{xy} - \lambda \mathbf{I}) = 0 \).

Clearly if \( \lambda = 1 \) then this becomes \( \det \begin{pmatrix}
\cos\theta - 1 & -\sin\theta & 0 \\
\sin\theta & \cos\theta - 1 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0 \), which definitely is 0 (since one row and column are 0).

Let's treat \( \lambda \) as unknown:

\[
\det \begin{pmatrix}
\cos\theta - \lambda & -\sin\theta & 0 \\
\sin\theta & \cos\theta - \lambda & 0 \\
0 & 0 & 1 - \lambda
\end{pmatrix} = 0
\]

\[
= \lambda^3 - (\cos\theta - 1) \lambda^2 + (\cos\theta + \sin^2\theta) \lambda - 1
\]

This equation has roots: \( \lambda = 1, \cos\theta + \sin^2\theta = 1, e^{\pm i\theta} \) \(_{\text{expected}}\) \( \pm 1 \) since \( \theta \neq \pi \).

Let's find the eigenvectors:

\[
\lambda_1 = 1 \quad R_{xy} \mathbf{x}_1 = \begin{pmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = b = 0 \quad c = \text{anything}
\]

\[
\lambda_2 = e^{i\theta} \quad R_{xy} \mathbf{x}_2 = \begin{pmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} e^{i\theta} \\ e^{i\theta} \\ e^{i\theta} \end{pmatrix} \Rightarrow a \cos\theta - b \sin\theta = a \cos\theta + b \sin\theta \\
a \sin\theta + b \cos\theta = b \cos\theta + b \sin\theta \\
c = c \cos\theta + c \sin\theta
\]

\[
-2b = a; \\
a = b; \\
c = 0
\]

\[
\lambda_3 = e^{-i\theta} \quad R_{xy} \mathbf{x}_3 = \begin{pmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} e^{-i\theta} \\ e^{-i\theta} \\ e^{-i\theta} \end{pmatrix} \Rightarrow a \cos\theta - b \sin\theta = a \cos\theta - b \sin\theta \\
a \sin\theta + b \cos\theta = b \cos\theta - b \sin\theta \\
c = c \cos\theta - c \sin\theta
\]

\[
 b = a; \\
a = b; \\
c = 0
\]

So \( \mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \), \( \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \), and \( \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} \).