Recall the definition of adjoint matrices: $A$ with elements $a_{ij} \rightarrow A^t$ with elements $(A^t)_{ij} = a_{ji}^*$.

or just $A^t = A^*$

s.t. $A^t \cdot B = B \cdot A^t$ and $(A^t)^t = A$.

Now, let us instead define "adjoint" more abstractly in terms of linear operators.

Let $A$ be a linear transformation on a vector space $U$.

For every $A$ be operator $A^*$ s.t. $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for every $x, y \in U$, is called the adjoint.

or equivalently $\langle x, A y \rangle = \langle A^* x, y \rangle$ since $\langle x, A y \rangle = \overline{\langle A^* x, y \rangle}$.

Useful things about adjoints are that given $A$, $A^*$ always exists and is unique, and $A^t$ itself is a linear operator (all provable).

With this operator/inner product definition we also find:

1. $(A + B)^t = A^t + B^t$
2. $(AB)^t = B^t A^t$
3. $(\alpha A)^t = \overline{\alpha} A^t$ w/ $\alpha$ a scalar
4. $(A^t)^t = A$

Let's prove (1) just to show that we do not need matrix properties to do so.

Since $\langle x, \overline{y} \rangle = \langle x, y \rangle$ where $C$ is a linear operator.

If $C = A + B$ then $\langle x, [A + B]^t y \rangle = \langle [A + B] x, y \rangle = \langle (A + B)x, y \rangle$ using linearity of operators

$= \langle A x + B x, y \rangle$ using linearity of inner product

$= \langle Ax, y \rangle + \langle B x, y \rangle$ using definition of adjoint

$= \langle x, A^* y \rangle + \langle x, B^* y \rangle$ using linearity of inner product

$= \langle x, [A + B]^t y \rangle$ using linearity of operators

Now does this mean $(A + B)^t = A^t + B^t$? Well suppose we had $\langle x, A y \rangle = \langle x, B y \rangle$. Does this $\Rightarrow A = B$?

The answer seems to be yes, but suppose that $x$ or $y$ is $0$. Then no!

In fact consider if $A$ and $B$ take $y$ and project them to different subspaces which are orthogonal to $x$. Then no!

We can save it by saying "if all of this holds for all values of $x$ and $y$" because then we have the theorem: if $\langle x, Ay \rangle = 0$ for all $x$ and $y \Rightarrow A = 0$, which applied to $\langle x, Ay \rangle = \langle x, y \rangle = 0 \Rightarrow (x, A^* y) = 0$.

Similarly to prove (2):

$\langle x, (AB)^t y \rangle = \langle AB x, y \rangle = \langle B x, A y \rangle = \langle x, B^t A^* y \rangle$ for all $x$ and $y$ (we theorem on $\langle x, A y \rangle = 0$)
Clearly, due to the isomorphism between matrices and linear operators, there should be a connection between the separate definitions of adjoint. There is!

Let the matrix of \( A \) have components \( a_{ij} \) w.r.t. an orthonormal basis \( \{x_i\}_{i=1}^n \). Then the matrix \( A^\dagger \) w.r.t. \( \{x_i\}_{i=1}^n \) is \( [A^\dagger]_{ij} = a_{ji}^* \) (which was our matrix definition).

To prove the connection, start with the matrix elements of a linear transformation w.r.t. the orthonormal basis \( x = \{x_i\}_{i=1}^n \), i.e., \( a_{ij} = \langle x_j, A x_i \rangle \). Then \( a_{ij} = \langle A^\dagger x_j, x_i \rangle = \langle x_j, (A^\dagger)^* x_i \rangle = [A^\dagger]_{ji}^* \Rightarrow A^\dagger_{ij} = a_{ji}^* \) (operator adjoint).

Now with the definition of adjoint in hand, we can specify a special class of linear operators.

If \( A = A^\dagger \), then \( A \) is "self-adjoint".

\( \begin{cases} \text{Real inner-product space } \Rightarrow \text{adjoint symmetric } (A = A^\dagger) \\ \text{Complex inner-product space } \Rightarrow \text{adjoint } \text{Hermitian} \end{cases} \)

I know you have worked with Hermitian operators/transformations in \( \mathbb{C} \mathbb{H} \), and we familiarized some of their properties, e.g., real eigenvalues. Let's explore more mathematically.

If \( A \) and \( B \) are self-adjoint \( \Rightarrow \) so is \( A + B \). (Obvious) \( (A + B)^\dagger = A^\dagger + B^\dagger = A + B = (A + B) \)

If \( A \) is self-adjoint \( \Rightarrow \) so is \( \alpha A \) for real \( \alpha \). (Obvious) \( [\alpha A]^\dagger = \alpha^* A^\dagger = \alpha^* A = \alpha A \) if \( \alpha \) real.

If \( A \) and \( B \) are self-adjoint \( \Rightarrow \) \( AB \) is self-adjoint iff \( [AB]^\dagger = AB \).

To prove the last one (which is less than obvious):

\[ (AB)^\dagger = B^\dagger A^\dagger \Rightarrow \langle B^\dagger A^\dagger x, y \rangle = \langle x, (AB)^\dagger y \rangle \]

\[ = \langle x, (BA)^\dagger y \rangle = \langle x, (BA) y \rangle \Rightarrow \langle B^\dagger A^\dagger x, y \rangle = \langle x, AB y \rangle \]

\[ \Rightarrow (AB)^\dagger = BA \]
Recall the theorem:

(i) A linear transformation $A$ on an inner-product space is $0$ if and only if $(x, Ay) = 0$ for all $x, y$.

It turns out that in certain situations this can be "strengthened".

(ii) If $A$ is a self-adjoint transformation on a real inner-product space, then $A = 0$ if and only if $(x, Ax) = 0$ for all $x$.

(iii) If $A$ is any linear transformation in a complex inner-product space, then $A = 0$ if and only if $(x, Ax) = 0$ for all $x$.

Before proving them, let's consider their "strength." Strength is tied to the condition that must be met. For all $x$ and $y$, means that $x$ and $y$ may differ, whereas for all $x$ in the latter two means you pair $x$ to itself, but need not worry about when $x \neq y$.

(iii) To prove necessity in all of these, note that if $A = 0$, $(x, Ay) = 0$ for all $x$ and $y$ including $x = y$.

To prove sufficiency, start with the first:

(i) If $(x, Ay) = 0$ for all $x$ and $y$, then if we take $x = Ay$, $(A^*y, Ay) = 0 \Rightarrow A^* = 0$ for all $y$ and therefore $A = 0$.

(ii) For the second, we'll show that the condition to be met actually reduces to that of the first.

If $(x, Ax) = 0$ for all $x$, then consider $(x + y, A^*(x + y)) = 0 = (x, Ax) + (x, Ay) + (y, Ax) + (y, Ay)$

which gives $t(x, Ay) - (x, Ax) - (y, Ay) = (x, A^*y) + (y, Ax) = 0$ for all $x$ and $y$.

but $(y, Ay) = (A^*x, y) = (A^*x, y) = (x, A^*y) = (x, Ay)$

then $(x, Ay) = 0$ for all $x$ and $y$ and we have already shown $A = 0$.

(iii) For the third we can actually use part of the proof of the second.

If $(x, Ax) = 0$ for all $x$, then following (iii) $(x, A^*y) = 0$ for all $x$ and $y$.

This time, since things can be imaginary, let's take $y = iy$, then $(x, A^*y) = (y, Ay) = 0$ for all $y$.

Then $(x, Ay) = x^* \overline{y} = (x, y^*) = (x, A^*y) = (x, Ay) = 0$ for all $x$ and $y$.

Adding this to earlier we have $2(x, Ay) = 0$ for all $x$ and $y \Rightarrow A = 0$.

Obviously we can combine the last two theorems into the statement:

If $A$ is self-adjoint (on real or complex $V$) then $A = 0$ if and only if $(x, Ax) = 0$ for all $x$. 

\[ \text{Lecture6–Inner Product Weighs in on Determinant and Self-Adjoint Operators Page 3} \]
Now why all of this? Well vector spaces are used all over physics, but if you think about it, using a complex vector space seems to be less applicable since all physical quantities (things we measure) are real valued. But hopefully you are starting to see that some of the math is even more powerful when extended to a unitary space. For example theorem (ii) works for self-adjoints, while theorem (iii) works for any.

So is there a way out of needing the advantages of complexity, but being restricted to real measurable? The answer is yes! And it relies on equating physical measurable to Hermitian operators/transformations.

Here's two wonderful results:

A linear transformation \( A \), on a unitary space, is Hermitian if and only if \( (x, Ax) \) is real for all \( x \).

This lets us safely (and usefully) define a measured quantity w.r.t. \( (x, Ax) \). What is it? It's the expectation value of \( A \) w.r.t. the state identified by \( x \). That is, it is the average value of \( A \) obtained over many measurements (so long as \( x \) is normalized).

Well that's the average value of \( A \). What about the possible results on single measurements?

Well...

The eigenvalues of a Hermitian operator/transformation are real.

These are them!!

These play such an important role for opening the connection between complex vector spaces and complex operators to real quantities which is the mathematical backbone of QM. Let's prove.

For the first:

If: If \( A = A^\dagger \) then \( (x, Ax) = (A^\dagger x, x) = (Ax, x) \) is real since \( (\cdot, \cdot)^\dagger \) and only if: If \( (x, Ax)^\dagger = (Ax, x) = (x, A^\dagger x) \) (note we haven't used \( A = A^\dagger \))

Then \( (x, [A - A^\dagger] x) = 0 \) for all \( x \) \( \Rightarrow A - A^\dagger = 0 \) \( \Rightarrow A = A^\dagger \)

And the second:

\( Ax = \lambda x \Rightarrow (x, Ax) = (x, \lambda x) = \lambda \|x\|^2 \Rightarrow \lambda = \frac{(x, Ax)}{\|x\|^2} \) but \( (x, Ax) \) is real by the previous as is \( \|x\|^2 \).
Another subgroup of linear transformations which is extremely useful are called “isometries.”

For a linear transformation $U$ on an inner product space $V$, we define:

If $V$ is complex and $U^*U = UU^* = I$ then $U$ is “unitary.”

If $V$ is real and $\hat{U}U = U\hat{U} = I$ then $U$ is “orthogonal.”

$U$ is an isometry.

Note that imply: $(U^*)^* = U = (U^*)^{-1} = (U^{-1})^*$ and similarly for $~$

Now this definition doesn’t really use the inner-product, but it turns out that these transformations are particularly important for inner-product spaces. In the following, if one is true, all are true.

$U$ on an inner product space satisfies:

1. $U^*U = I$

2. $(Ux, Uy) = (x, y)$ for all $x$ and $y$

3. $\|Ux\| = \|x\|$ for all $x$

To prove $1 \Rightarrow 2$, start with $U^*U = I$ being true:

$\langle U^*Ux, U^*y \rangle = \langle U^*y, x \rangle = (x, y)$ for all $x$ and $y$.

And if $x = y$, $\langle Xu, Uy \rangle = \|Ux\|^2 = (x, x) = \|x\|^2$ for all $x$.

So $1 \Rightarrow 2 \Rightarrow 3$, and to finish up we need $3 \Rightarrow 1$

If $\|Ux\| = \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle = (x, x) = \|x\|^2$ for all $x$.

Recall that if $(A x, x) = 0$ for all $x$ and $A$ is self-adjoint then $A = 0$.

So if $[U^*U - I]$ is self-adjoint then $U^*U - I = 0 \Rightarrow U^*U = I$ and 1 holds.

Is $[U^*U - I]$ self-adjoint? $[U^*U - I]^* = (U^*)^* - I^* = UU^* - I = U^*U - I$ so yes it is!

So isometries preserve lengths of vectors, but they also preserve the angle between two vectors:

$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|} \Rightarrow \frac{\langle Ux, Uy \rangle}{\|Ux\| \|Uy\|} = \frac{\langle x, y \rangle}{\|x\| \|y\|} = \cos \theta$

But the preservation of lengths and angles means that an isometry will carry one orthonormal set into another. The only concern might be whether it takes a complete orthonormal basis into another, and this can be proven using Persius’s equation.

So we have: If $\{x: 3\}$ is a complete orthonormal basis, then so is $\{Ux: 3\}$ for any isometry $U$. 

Lecture7-Isometries, the BIG Picture and the Normal Picture Page 1
Going back to eigenvalues, we can draw the results for self-adjoint and isometric transformations:

For Hermitian transformations we know the eigenvalues are all real. But the same is true for symmetric transformations. Which leads to the slightly more general:

If $A$ is a self-adjoint transformation, then all of its eigenvalues are real.

To prove this recall eigenvalue $\lambda$ of $A$ or $i.e. A x = \lambda x$ for $x \neq 0$.

Now $\langle x, A x \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle$, but if $A = A^*$ then $\langle x, A x \rangle = \langle A^* x, x \rangle = \langle A x, x \rangle = \lambda^* \langle x, x \rangle$

So we have $\lambda \langle x, x \rangle = \lambda^* \langle x, x \rangle$ and $\langle x, x \rangle \neq 0$ so $\lambda = \lambda^*$.

For isometric transformations we have:

All eigenvalues of isometric transformations have absolute values of 1.

Proof:

If $U$ is an isometry and $U x = \lambda x$ for $x \neq 0$ then $\|x\| = \|U x\| = |\lambda| |x| \Rightarrow |\lambda| = 1$

since $\|x\| \neq 0$. 