Now how does having/using an orthonormal basis impact determinants?

Consider a matrix $A = \begin{pmatrix} a_1 & a_2 & \cdots \end{pmatrix}$ where each row is a vector orthogonal to the rest of the rows. So we might call it $A = \begin{pmatrix} a_1 \end{pmatrix}$ where each $a_i$ has a component (as according to book $A = \begin{pmatrix} a_1, a_2, \cdots \end{pmatrix}$)

Clearly $AA^\top = \begin{pmatrix} \sum_j a_j a_j^\top \end{pmatrix} = \begin{pmatrix} a_1 a_1^\top & a_2 a_2^\top & \cdots \end{pmatrix} \begin{pmatrix} a_1^\top & a_2^\top & \cdots \end{pmatrix}$

This just takes each row and forms the inner product with all the others.

So $AA^\top = \begin{pmatrix} \|a_1\|^2 & 0 & 0 \\ 0 & \|a_2\|^2 & 0 \\ 0 & 0 & \|a_3\|^2 \end{pmatrix} \Rightarrow \det(AA^\top) = \|a_1\|^2 \|a_2\|^2 \|a_3\|^2$

Recall that $\det A = (\det A^\top)^* \Rightarrow \det A^* = \det A^\top$ and $\det(AB) = \det A \det B$

Together there gives $\det(AA^\top) = \det A \det A = \det A \det A^\top = \det A^\top \det A \Rightarrow \det A = \sqrt{\det A^\top \det A} = \|a_1\| \|a_2\| \|a_3\|$

Now $A$ in the above was special. Suppose we start with an arbitrary matrix $B = \begin{pmatrix} b_1 & b_2 & \cdots \end{pmatrix}$

This time the rows of the matrix need not be orthogonal. Now let's Gram-Schmidt the shit out of it: $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} \Rightarrow \hat{B} = \begin{pmatrix} b_1 \\ b_2 - \frac{\langle b_2, b_1 \rangle}{\|b_1\|^2} b_1 \\ \vdots \end{pmatrix}$

Now let's form $C = \begin{pmatrix} \|b_1\| b_1' \\ \|b_2\| b_2' \langle b_2, b_1 \rangle \|b_1\| b_1' \\ \vdots \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - \frac{\langle b_2, b_1 \rangle}{\|b_1\|^2} b_1 \\ \vdots \end{pmatrix}$

This is special, since each row is formed by adding multiples of other rows to it, e.g., $b_2 \rightarrow b_2 - \frac{\langle b_2, b_1 \rangle}{\|b_1\|^2} b_1$

But this means $\det \hat{B} = \det C$!

But since $C$ is comprised of orthonormal vectors $\{c_1, c_2, \cdots, c_n\}$, inner scalar multiples $c_1, b_2, \cdots, b_n \Rightarrow \|b_1\|, \|b_2\| - \frac{\langle b_2, b_1 \rangle}{\|b_1\|^2} \|b_1\|, \cdots$, then $\det C = \|b_1\| \|b_2\| \cdots \|b_n\|$

where each $\|b_i\| \leq \|b_i\|$, therefore we get Hadamard's inequality: $|\det B| \leq \|b_1\| \|b_2\| \cdots \|b_n\|$

And finally we end by stating what can be obvious, but will nonetheless be useful later on:

A linear transformation $A$ on an inner-product space is the zero transformation if and only if $\langle x, Ay \rangle = 0$ for all $x$ and $y$. 

Lecture8-Self-adjoint is Symmetric or Hermitian Page 1
Recall the definition of adjoint matrices: \( A \) with elements \( a_{ij} \) goes to \( A^+ \) with elements \( (A^+)^*_{ij} = a_{ji}^* \) or just \( A^+ = A^* \) s.t.
\[
\begin{align*}
(A^+)^* &= A \quad \text{or} \quad (A^*)^* = A \\
(A + B)^* &= A^* + B^* \\
(A^*B^*)^* &= B^*A^*
\end{align*}
\]

Now let us instead define “adjoint” more abstractly in terms of linear operators.

Let \( A \) be a linear transformation on a vector space \( V \).

For every \( A \) the operator \( A^+ \) s.t. \( (Ax, y) = (x, A^+y) \) for every \( x, y \in V \) is called the adjoint operator.

or equivalently \( (x, A^+y) = (A^*x, y) \) since \( (x, A^+y) = (A^*x, y) = (y, A^*x)^* = (y, A^*x) = (x, A^*y) \)

Useful things about adjoints are that given \( A \), \( A^* \) always exists and is unique, and \( A^* \) itself is a linear operator (all provable).

With this operator/inner product definition we also find:

1. \( (A + B)^* = A^* + B^* \)
2. \( (AB)^* = B^*A^* \)
3. \( (\alpha A)^* = \alpha^* A^* \) \( \alpha \) a scalar
4. \( (A^*)^* = A \)

Let’s prove (4) just to show that we do not need matrix properties to do so.

Since \( (x, C^*y) = (Cx, y) \) where \( C \) is a linear operator.

If \( C = A + B \) then \( (x, [A + B]^*y) = (A + B)x, y) = (A + B)^*x, y) \) using linearity of operators

\[
= (A^*x, y) + (B^*x, y) \quad \text{using linearity of inner product}
\]

\[
= (x, A^*y) + (x, B^*y) \quad \text{using definition of adjoint}
\]

\[
= (x, A^*y) + (B^*x, y) \quad \text{using linearity of inner product}
\]

\[
= (x, [A^* + B^*]y) \quad \text{using linearity of operators}
\]

Now does this mean \( [A + B]^* = A^* + B^* ? \) We suppose we had \( (x, A^*y) = (x, B^*y) \). Does this \( \Rightarrow A = B \)?

The answer seems to be yes, but suppose that \( x \) or \( y \) is 0. Then no!

In fact consider if \( A \) and \( B \) take \( y \) and project them to different subspaces which are orthogonal to \( x \). Then no!

We can save it by saying "if all of this holds for all values of \( x \) and \( y \)" because then we have the theorem: \( \langle x, A^*y \rangle = 0 \) for all \( x \) and \( y \) \( \Rightarrow A^* = 0 \), which applied to \( (x, A^*y) = (x, B^*y) = 0 \Rightarrow (y, A^*x) = 0 \) similarly to prove (2):

\[
\langle x, (AB)^*y \rangle = \langle ABx, y \rangle = \langle Bx, A^*y \rangle = \langle x, B^*A^*y \rangle \quad \text{for all} \ x \ and \ y \ (\text{w/ theorem on} \ x, A^*y = 0)
\]
Clearly, due to the isomorphism between matrices and linear operators, there should be a connection between the separate definitions of adjoint. There is!

Let the matrix of $A$ have components $a_{ij}$ w.r.t. an orthogonal basis $x$. Then the matrix $A^\dagger$ w.r.t. $x$ is $[A^\dagger]_{ij} = a_{ji}$, (which was our matrix definition).

To prove the connection, start the matrix elements of a linear transformation w.r.t. the orthonormal basis $x = \{x_i, y_j \}$, i.e., $a_{ij} = \langle x_i, A y_j \rangle$. Then $a_{ij} = \langle A^\dagger x_i, y_j \rangle = \langle x_i, (A^\dagger)^\dagger y_j \rangle = \langle x_i, A x_j \rangle^* \Rightarrow A^\dagger y_j = a_{ji}^* \langle x_i, \rangle \langle \rangle^*$ of adjoint.

Now with the definition of adjoint in hand, we can specify a special class of linear operators.

- If $A = A^\dagger$, then $A$ is "self-adjoint." (Real inner-product space $\Rightarrow$ adjoint symmetric ($A^\dagger = A$))
- Complex inner-product space $\Rightarrow$ adjoint is Hermitian.

I know you have worked with Hermitian operators/transformations in QM, and we familiar with some of their properties, e.g., real eigenvalues. So let's explore more mathematically.

- If $A$ and $B$ are self-adjoint, so is $A + B$. (Obvious)\[ (A + B)^\dagger = A^\dagger + B^\dagger = A + B \]
- If $A$ is self-adjoint so is $\alpha A$ for real $\alpha$. (Obvious)\[ \alpha^* A^\dagger = \alpha^* A = \alpha A \iff \alpha \in \mathbb{R} \]
- If $A$ and $B$ are self-adjoint $AB$ is self-adjoint iff $[A, B] = 0.$

To prove the last one (which is less obvious):

If $[A, B] = 0$, $AB = BA \Rightarrow (AB)^\dagger = B^\dagger A^\dagger = BA = AB$

Only if $[A, B] = 0$, $AB = (AB)^\dagger = B^\dagger A^\dagger = BA$
Recall the theorem:

(i) A linear transformation $A$ on an inner-product space is 0 if and only if $(x,Ay) = 0$ for all $x,y$.

It turns out that in certain situations this can be strengthened:

(ii) If $A$ is a self-adjoint in a real inner-product space, then $A = 0$ iff $(x,Ax) = 0$ for all $x$, and $A = \mathbf{I}$ is unitary.

(iii) If $A$ is any linear transformation in a complex inner-product space, then $A = 0$ iff $(x,Ax) = 0$ for all $x$.

Before proving them, let's consider their "strength." Strength is tied to the condition that must be met. For all $x$ and $y$, means that $x$ and $y$ are distinct, whereas for all $x$ in the latter two means you pick $x$ to itself, but need not worry about when $x \neq y$.

To prove necessity in all of these, note that if $A = 0 \Rightarrow (x,Ay) = 0$ for all $x$ and $y$ including $x = y$.

To prove sufficiency, start with the first:

If $(x,Ax) = 0$ for all $x$ and $y$, then if we take $x = Ay$, $(Ax,Ay) = 0 \Rightarrow A = 0$ for all $y$ and therefore $A = 0$.

For the second, we'll show that the condition to be met actually reduces to that of the first.

If $(x,Ax) = 0$ for all $x$ then consider $<x+y,A[x+y]> = 0 = (x,Ax) + (x,Ay) + (y,Ax) + (y,Ay)$ which gives $<x+y,A[x+y]> - <x,Ax> - <y,Ay> = (x,Ay) + (y,Ax) = 0$ for all $x$ and $y$.

but $(y,Ax) = (Ax,y)^* = (Ax,y) = (x,A^*y) = (x, Ay)$

then $<x, Ay> = 0$ for all $x, y$, but we have already shown $A = 0$.

For the third we can actually use part of the proof of the second.

If $(x,Ax) = 0$ for all $x$ then following (ii) $(x,Ay) + (y,Ax) = 0$ for all $x$ and $y$.

This time, since things can be imaginary, let's take $y = iy$, then $(x,Aiy) + (iy,Ax) = 0$ for all $x$ and $y$.

Then $<x, Ay> = <x,y> A (x,y)$ we have $<x,Ay> = (y,Ax)$ and $(x,Ay) = (x,A^*y) = (x,Ay) = 0$.

Adding this to earlier we have $<x, Ay> = 0$ for all $x \neq y \Rightarrow A = 0$.

Obvioulsy we can combine the last two theorems into the statement:

If $A$ is self-adjoint (on real or complex $V$) then $A = 0$ iff $(x,Ax) = 0$ for all $x$.