1. Consider the Lagrangian for QCD based on SU(3):

$$\mathcal{L} = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} m^2 \Phi^2 + \frac{i}{2} \bar{u} \gamma ^{\mu} D_\mu u - \frac{i}{2} \bar{u} \gamma ^{\mu} u A_\mu \gamma ^5$$

Which is invariant under:

$$\Phi \rightarrow e^{i \alpha} \Phi, \quad \bar{u} \xrightarrow{A_\mu} \bar{u} = e^{-i \lambda \mathcal{A}} \bar{u} e^{i \lambda \mathcal{A}}$$

For local transformation parameters $\phi(x)$.

If we instead restrict to an abelian group then the Lie algebra of generators becomes $E_8, \mathcal{A} = 0 \Rightarrow e^{0} = 0$.

We can immediately see that the gauge field kinetic term reduces to $\frac{1}{2} \mathcal{F}_{\mu \nu} \mathcal{F}^{\mu \nu}$ which is what we expect for an abelian symmetry, e.g. $\mathcal{U}(1)$ of $E_{8}$/SO(8) (or even something like $\mathcal{U}(1) \times \mathcal{U}(1)$).

As for the gauge field transformation, since all of the $\lambda$s commute, there is no problem moving the exponentials around, i.e.

$$\lambda \mathcal{A} = e^{-\lambda \mathcal{A}} \lambda \mathcal{A} e^{\lambda \mathcal{A}}$$

This is okay if all the $\lambda$s commute. Normally you would have to be very careful taking this derivative. For example would you write $\lambda (\mathcal{A} - \mathcal{A} x) e^{\lambda \mathcal{A}}$ or $e^{\lambda \mathcal{A}} \lambda \mathcal{A} e^{-\lambda \mathcal{A}}$? Fortunately, for the abelian case it does not matter since they are the same!

Thus:

$$\lambda \mathcal{A} \rightarrow \lambda \mathcal{A} + \frac{i}{2} \mathcal{D}_\mu (-i \gamma ^\mu \lambda \mathcal{A})$$

having cancelled all the exponentials

$$= \lambda \mathcal{A} + \lambda \mathcal{D} \lambda$$

If we only have one generator this becomes $\mathcal{A}' = \mathcal{A} + \mathcal{D} \lambda$ as expected.
2. \[ z = \frac{1}{2} \left( \frac{\partial}{\partial t} \right) \phi + \frac{i}{2} \left( \lambda^i \right) \phi \] where \[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \]

(a) The Lagrangian is invariant under a global symmetry that transforms \( \phi \to \phi' = \gamma \phi \), \( \gamma^T \phi = -\gamma \phi \) as long as \( \gamma^T \phi \to \gamma^T \phi' = \gamma \phi \). Since \( \gamma^T \gamma = 1 \), we are dealing with \( SO(3) \).

Note: \( \gamma^T = \gamma \) defines \( O(3) \), but we also need \( det(\gamma) = 1 \) so that \( \gamma \) is a Lie group.

As usual we can write \( \gamma = e^{ig_L A} \) where \( A = (A^1, A^2, A^3) \) are the generators of \( SO(3) \) and \( [A^i, A^j] = i \epsilon^{ijk} A^k \) and \( \phi = (\phi_1, \phi_2, \phi_3) \) is a vector of parameters

(b) \( \partial_\phi \phi = \partial_\phi \phi + ig_L A \phi \)

For invariance we need \( \partial_\phi = \partial_\phi' = e^{ig_L A} \partial_\phi \)

Then: \( \partial_\phi' = \partial_\phi + ig_L A \phi' = \partial_\phi (e^{ig_L A}) + ig_L A (e^{ig_L A}) \)

\[ = \partial_\phi \left( e^{g_L A} \right) + e^{g_L A} \partial_\phi + ig_L A \phi \]

We want: \( e^{g_L A} \left( \partial_\phi + ig_L A \phi \right) \)

Which we get: \( A = e^{g_L A} \left( \partial_\phi + ig_L A \phi \right) \)

Now the derivative term in \( \phi \) should be understood as \( \frac{1}{2} (g_L^T g_L) \partial \phi \) and we now have:

\[ \frac{1}{2} (g_L^T g_L) \partial \phi \to \frac{1}{2} (g_L^T g_L) \partial \phi' = \frac{1}{2} (e^{g_L A})^T (g_L^T g_L) e^{g_L A} \partial \phi = \frac{1}{2} (g_L^T g_L) e^{g_L A} \partial \phi \]

But for \( SO(3) \) the generators satisfy \( A^T = -A \) so this gives: \( \frac{1}{2} (g_L^T g_L) e^{g_L A} \partial \phi = \frac{1}{2} (g_L^T g_L) D \phi \)

(c) \[ F_{\mu \nu} = \frac{1}{2} \left( \partial_\mu \phi_\nu - \partial_\nu \phi_\mu + ig_L A \phi_\mu A \phi_\nu \right) \]

\[ = \frac{1}{2} \left[ \partial_\mu \phi_\nu + ig_L A \phi_\mu A \phi_\nu + ig_L A \phi_\nu A \phi_\mu \right] - \partial_\nu \phi_\mu + ig_L A \phi_\mu \phi_\nu \]

\[ = \frac{1}{2} \left[ ig_L \left( \delta_\mu^\nu \phi + ig_L A \phi_\mu A \phi_\nu - g_L \lambda^i A_i^\mu \phi_\nu \right) + ig_L \left( \delta_\nu^\mu \phi + ig_L A \phi_\mu A \phi_\nu - g_L \lambda^i A_i^\nu \phi_\mu \right) \right] - \partial_\nu \phi_\mu + ig_L A \phi_\mu \phi_\nu \]

\[ = \partial_\mu \phi_\nu + ig_L \lambda^i A_i^\mu A \phi_\nu + ig_L \lambda^i A_i^\nu A \phi_\mu \]

\[ = O(\phi) - \partial_\mu \phi_\nu + ig_L \lambda^i A_i^\mu A \phi_\nu + ig_L \lambda^i A_i^\nu A \phi_\mu \]

\[ = \lambda^i \left( - \partial_\mu A_i^\nu + \partial_\nu A_i^\mu + ig \left[ \lambda^j, \lambda^k \right] A_i^\mu A_i^\nu \right) \]

\[ = \lambda^i \left( - \partial_\mu A_i^\nu + \partial_\nu A_i^\mu + ig \left[ \lambda^j, \lambda^k \right] A_i^\mu A_i^\nu \right) = \lambda^i F_{\mu \nu} \]

Then adding \( \frac{i}{16} \sum_k F_{\mu \nu} F_{\mu \nu} \) to \( \phi \) will allow \( A \mu \phi \) to propagate.

(d)
3. 1) $I = \left( X R Y^2 O X R + \kappa c X L Y^2 O X L + \frac{\kappa d}{\gamma} X R X L + \frac{\kappa c}{\gamma} X L X \right)$

-tens arising L and R vanish due to $\gamma$
-tens with some RR or LL vanish automatically

**But:** Since $\delta_u \chi_{\mu} \times \delta_s \chi_{\nu}$ allows independent transformations of $\chi_L$ and $\chi_R$, there is no way the wave fronts are invariant under all elements of $\delta_u \chi_{\mu} \times \delta_s \chi_{\nu}$.

So we start with:

$I = k_c X R Y^2 O X R + k_c X L Y^2 O X L$

where $\delta_s \chi_{\mu}$ acts on $\chi_L$ as $e^{-i g_\delta \chi_{\mu} \delta} \chi L$ and $\delta_u \chi_{\nu}$ acts on $\chi_L$ as $e^{i g_\delta \chi_{\nu} \delta} \chi L$

Notice that $\delta_s \chi_{\mu}$ since there are two different groups, $\delta$ is the same in each case since each is $\delta_u \chi_{\nu}$.

5) To gauge we replace $\partial_\mu \chi_L \rightarrow \omega_\mu \chi_L = \omega_\mu \chi_L + i g_\delta \delta \chi L \omega_\mu \chi_L$

Note: 6 new gauge fields, 3 $\omega_{\mu L}$ and 3 $\omega_{\nu R}$

To have $\partial_\mu \chi_L \rightarrow \partial' \mu \chi_L = e^{-i g_\delta \delta \chi L \omega_\mu \chi_L}$ $\partial_\mu \chi_L$ we need:

$\partial' \mu \chi_L = \partial_\mu \chi_L + i g_\delta \delta \chi L \omega_\mu \chi_L = \partial_\mu (e^{-i g_\delta \delta \chi L \omega_\mu \chi_L} + i g_\delta \delta \chi L \omega_\mu \chi_L e^{-i g_\delta \delta \chi L \omega_\mu \chi_L})$

which we want to be $e^{-i g_\delta \delta \chi L \omega_\mu \chi_L}$ $\partial_\mu \chi_L + e^{-i g_\delta \delta \chi L \omega_\mu \chi_L} i g_\delta \delta \chi L \omega_\mu \chi_L$

$\omega_\mu \chi L$ we need $\delta \delta \chi L \omega_\mu \chi_L = e^{-i g_\delta \delta \chi L \omega_\mu \chi_L} \delta \delta \chi L \omega_\mu \chi_L + \frac{i}{2} \partial_\mu (e^{-i g_\delta \delta \chi L \omega_\mu \chi_L}) e^{-i g_\delta \delta \chi L \omega_\mu \chi_L}$

and similarly for $\delta \delta \chi L \omega_\mu \chi_L$.

6) To allow both sets of gauge fields to propagate we add gauge kinetic terms of the form:

$-\frac{i}{2} \partial_\mu F_{\lambda \nu}^{\alpha} F_{\alpha \nu}^{\lambda}$

where $F_{\lambda \nu}^{\alpha} = \partial_\lambda \chi_{\alpha} - \partial_\nu \chi_{\alpha} - g_\delta \delta \chi_{\alpha} \omega_{\nu}$

and similarly for $F_{\lambda \nu}^{\lambda}$.