Particle Physics HW7

1. As a follow up from the gauging story, I now want you to demonstrate that the non-abelian gauge field kinetic term \( \frac{1}{16\pi} F_{\mu\nu}^a F^{a\mu\nu} \) is itself gauge invariant. To do so it is easier to call the transformations \( e^{ig\lambda(x)} \) which are in general a set of non-commuting matrices (since the \( \lambda \) are) by \( U(x) \) and just keep in mind that \( U(x) \) are non-commuting matrices. Also, you can work with the full set of gauge fields all at once by using \( A_\mu \equiv \lambda^a A^a_\mu \) and hence work in terms of \( F_{\mu\nu} \equiv \lambda^a F^a_{\mu\nu} \). Again as long as you keep in mind that \( A_\mu \) and \( F_{\mu\nu} \) are non-commuting objects.
   a) Write the gauge field transformation law for \( A_\mu \) in terms of the matrices \( U(x) \).
   b) Now determine how the gauge field strength \( F_{\mu\nu} \) will transform in terms of \( U(x) \).
   c) Finally consider \( F^a_{\mu\nu} F^{a\mu\nu} \) which in this language is just \( Tr(F_{\mu\nu} F^{\mu\nu}) \) where the trace is over the “color” space matrices. Demonstrate that this combination is invariant. It will help to recall that spacetime indices being upper or lower does not change anything with regards to gauge transformations, i.e. \( F_{\mu\nu} \) and \( F^{\mu\nu} \) will transform the same way. Hint: You will need to use the following (which you should prove), \( \partial_\mu (U^{-1}) = -U^{-1} \partial_\mu (U) U^{-1} \).
   d) Going back to your result from part (b), verify that if the group is actually abelian, then \( F_{\mu\nu} \) is invariant on its own.

2. For the gauged Higgs Lagrangian given in class, insert the expansion in small fluctuations about the nontrivial background background \( \phi_1 = \frac{1}{\sqrt{2} \lambda}, \phi_2 = \frac{1}{\sqrt{2} \lambda}, A_\mu = 0 \) to obtain the Lagrangian for the fluctuations \( (\eta, \beta, A_\mu) \). In particular, identify which terms in the final expression would correspond to kinetic, mass and more general interaction terms. You should see that the two scalar fluctuations both have mass in this case. However you should also see that there is a new interaction term which is only quadratic in the scalar fluctuations. The other interaction terms are cubic or higher. This interaction term is telling us that one scalar particle can freely interchange with the other, which means they are not physically distinguishable states. To get the physically distinguishable states, we should work with a linear combination for which this interaction is absent, e.g. the case done in lecture. This will be a lengthy but straightforward calculation.

3. Consider a Lagrangian for three real scalar fields \((\phi_1, \phi_2, \phi_3)\) of the form:
\[
L = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 + \frac{1}{2} \partial_\mu \phi_3 \partial^\mu \phi_3 - \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2 + \phi_3^2) + \frac{\lambda^2}{4} (\phi_1^2 + \phi_2^2 + \phi_3^2)^2
\]
If this three component scalar field is the Higgs field, determine the number and mass of the Higgs boson(s) as well as the number of Goldstone boson(s).