2. \( \phi_1 = \frac{x}{x} x, \phi_2 = \frac{y}{y} x, A_n = 0 \)

First we will do this by brute force. See next page for trick.

\[
\begin{align*}
\phi_1 &= \frac{x}{x} x + \frac{y}{y} x \nonumber \\
\phi_2 &= \frac{y}{y} x + \frac{z}{z} x \nonumber \\
A_n &= 0 + A_n \nonumber
\end{align*}
\]

\[
\begin{align*}
L_{\text{inv}}(e, \phi_1, A_n) &= \frac{1}{2} [ (\phi_2 - \phi_1 A_n) \phi_1^* ] [ (\phi_2 - \phi_1 A_n) \phi_2^* ] - \frac{1}{2} \phi_1^* \phi_1^* + \frac{i}{\hbar} \phi_1 \phi_2^* + \frac{i}{\hbar} \phi_2 \phi_1^* + \frac{i}{\hbar} \phi_0 \phi_0^* \\
N_{\text{inv}}: \quad \phi_0^* &= (\frac{x}{x} x + \frac{y}{y} x)^2 + (\frac{y}{y} x + \frac{z}{z} x)^2 = \frac{x^2}{x} x + \frac{y^2}{y} x + \frac{y^2}{y} x + x^2 + y^2
\end{align*}
\]

\[
L(n, \phi, A_n) = \frac{1}{2} [ (\phi_2 - \phi_1 A_n) \phi_1^* ] [ (\phi_2 - \phi_1 A_n) \phi_2^* ] - \frac{1}{2} \phi_1^* \phi_1^* + \frac{i}{\hbar} \phi_1 \phi_2^* + \frac{i}{\hbar} \phi_2 \phi_1^* + \frac{i}{\hbar} \phi_0 \phi_0^* + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}
\]

\begin{align*}
&= \frac{1}{2} (\phi_2 - \phi_1 A_n)(\phi_1 - \phi_0 A_n)(\phi_2 - \phi_1 A_n)(\phi_1 - \phi_0 A_n) \\
&+ \frac{1}{2} (\phi_2 - \phi_1 A_n)(\phi_1 - \phi_0 A_n)(\phi_2 - \phi_1 A_n)(\phi_1 - \phi_0 A_n)
\end{align*}

\[
\begin{align*}
- \phi_1^* - \phi_1 \phi_0 - \frac{y}{y} \phi_0^* - \frac{y}{y} \phi_0 \\
+ \frac{1}{\hbar} \phi_1^* + \frac{1}{\hbar} \phi_1 \phi_0^* + \frac{1}{\hbar} \phi_1 \phi_0 \\
&+ \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}
\end{align*}
\]

The underlined terms are important. The others are either a constant or standard interaction terms.

Then:

\[
L(n, \phi, A_n) = \frac{\hbar}{2} \phi_1^* \phi_1 + \frac{1}{2} \phi_2^* \phi_2^* - \frac{1}{2} \phi_1^* \phi_1 + \frac{i}{\hbar} \phi_1 \phi_2^* + \frac{i}{\hbar} \phi_2 \phi_1^* + \frac{i}{\hbar} \phi_0 \phi_0^* + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}
\]

Any time we see a quadratic term with \(2\) different fields (like the 3 examples above), this implies an interaction which takes one excitation, e.g. \(A^2\), and spontaneously changes it to another, e.g. \(B\). This may at first seem like a decay of \(A \rightarrow B\), but for true交流s there are always at least 3 states involved, e.g. \(A \rightarrow B + C\) ...

What these quadratic interactions are actually telling us is that \(B\) and \(A\) are not completely distinct, i.e., we are \(\pi\) and \(\pi^*\) or \(\pi\) and \(\overline{\pi}\).
To get the actual fundamental fields, let's consider a picture of what is going on.

We are expanding about the solution \( \Phi = \frac{1}{\sqrt{2}} \phi = \phi_0 + \epsilon \).

But our fluctuations are along the \( \phi, \phi_0 \) axes.

But from lectures we know the true massive Higgs boson should be a radial fluctuation, while the massless Goldstone mode should be along the circular valley.

To see this consider \( \phi' = \frac{1}{\sqrt{2}} (\pi + \delta) \Rightarrow \nu = \frac{1}{\sqrt{2}} (\pi' + \delta') \)

Now let's rewrite the important part of our previous result in terms of \( \pi' \) and \( \delta' \):

\[
\mathcal{L}(\pi, \delta, A_\mu) = \frac{1}{2} \partial_\mu \pi \partial^\mu \pi + \frac{1}{2} m^2 \pi^2 + \frac{1}{4} \epsilon \epsilon' \partial^\mu \partial_\mu (\pi' \delta' - \delta' \pi') + \frac{1}{4} \epsilon \epsilon' \partial_\mu \partial_\nu (\pi' \delta' - \delta' \pi')^2
\]

\[
\downarrow
\]

\[
\mathcal{L}(\pi', \delta', A_\mu) = \frac{1}{2} \partial_\mu \pi' \partial^\mu \pi' + \frac{1}{2} m^2 \pi'^2 - \frac{3}{2} \epsilon \epsilon' \partial_\mu \pi' \partial^\mu \delta' + \frac{3}{2} \epsilon \epsilon' \partial_\mu \pi' \partial^\mu \delta' - \frac{3}{2} \epsilon \epsilon' \partial_\mu \pi' \partial^\mu \delta' - \frac{3}{2} \epsilon \epsilon' \partial_\mu \pi' \partial^\mu \delta'
\]

On massless Higgs

On massless Goldstone

We still have the quadratic term \( \partial_\mu \delta' A_\mu \), but we know that we can use the original symmetry to "gauge away" fluctuations in \( \delta \) (set them to zero). This is where \( A_\mu \) into Goldstone.

**BTW:** Another route to getting the next Logaritam for \( \pi \) and \( \delta \) would be to start with the Lagrangian given in class for \( \pi \) and \( \delta \) about \( \Phi = \frac{1}{\sqrt{2}} \phi \), \( \Phi = \phi_0 \) and call that fluctuations \( \pi', \delta' \) then just substitute \( \pi' = \frac{1}{\sqrt{2}} (\pi + \delta) \), \( \delta' = \frac{1}{\sqrt{2}} (\pi - \delta) \) to get the Lagrangian on the previous page.
3. For \( L = \frac{1}{2} \partial \phi \cdot \partial \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \phi \partial^2 \Delta \phi \cdot \partial \phi - \frac{1}{4} \left( \phi_1^2 + \phi_2^2 + \phi_3^2 \right) + \frac{1}{4} \left( \phi_1 + \phi_2 + \phi_3 \right)^2 \), we notice that the symmetry group is transformations which leave \( \phi_1^2 + \phi_2^2 + \phi_3^2 \) constant. But these are just rotations in the 3D space of \( \phi_1, \phi_2, \phi_3 \). For fixed \( \phi_1, \phi_2, \phi_3 \), this means we are looking at the surface of a two-sphere, which means we should expect two Goldstone modes (compare with the circular symmetry from the example in class where we get one Goldstone mode).

Of course, we still have only one radially (or Higgs) mode. To get its mass we can just choose \( \phi_1 = \frac{\Delta}{2} \), \( \phi_2 = \phi_3 = 0 \) and then let \( \phi_1 \rightarrow \infty \) and search for quadratic terms in \( \phi_1 \).

From \( -\frac{\Delta^2}{4} \left( \phi_1^2 + \phi_2^2 + \phi_3^2 \right) \), we expect \( \frac{\Delta^2}{4} \phi_1^2 \).

From \( \frac{\Delta^2}{4} \left( \phi_1^2 + \phi_2^2 + \phi_3^2 \right) \), we expect \( \frac{\Delta^2}{4} \left( \phi_1^2 + \phi_2^2 + \phi_3^2 \right) \).

So in total the Higgs fluctuation has mass term \( \frac{\Delta^2}{4} \phi_1^2 \) and \( \frac{\Delta^2}{4} \phi_1^2 \), which gives

\[ m^2 = \frac{\Delta^2}{4} \phi_1^2. \]