You can try both problems below, but you will only receive credit for the most correct solution.

1. \((10pts)\) Show that any discrete group with 3 elements is essentially the same. Write down the (unique) multiplication table for a 3 element group.

   **Hint:** To enumerate all of the possibilities, you will need to assign an inverse to each element, and it turns out there are only two ways to do this. Then you can show that one of these leads to an inconsistency.

Any group must have an identity, so we consider \( \mathbb{E} = \{I, a, b\} \).

There are 2 possible inverse assignments:

\[
\begin{align*}
\text{(i)} & \quad a^{-1} = a, \quad b^{-1} = b \\
\text{(ii)} & \quad a^{-1} = b, \quad b^{-1} = a
\end{align*}
\]

For (i), we can consider possible values for \(ab\):

\[
\begin{align*}
ab = I & \quad \text{not allowed since then } a^{-1} = b \\
ab = a & \quad \text{not allowed since then } b = I \\
ab = b & \quad \text{not allowed since then } a = I
\end{align*}
\]

Case (i) is not consistent.

For (ii), we know \(ab = I\), but need values for \(aa\) and \(bb\):

\[
\begin{align*}
aa = I & \quad \text{not allowed since then } a = a^{-1} \\
aa = a & \quad \text{not allowed since then } a = I \\
ab = b & \quad \text{okay}
\end{align*}
\]

\[
\begin{align*}
bb = I & \quad \text{not allowed since then } b = b^{-1} \\
bb = b & \quad \text{not allowed since then } b = I \\
bb = b & \quad \text{okay again}
\end{align*}
\]

Then:

\[
\begin{array}{c|ccc}
\hline
& I & a & b \\
\hline
I & I & a & b \\
a & a & b & I \\
b & b & I & a \\
\hline
\end{array}
\]

Turn over for second problem!!
2) (10pts) For the SU(3) algebra $[g_i, g_j] = if^{ijk}g_k$ where

$$f^{123} = 1, \quad f^{147} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

and the $f^{ijk}$ are totally antisymmetric in the three indices, i.e. $f^{ijk} = -f^{jik}$ (if a particular index combination doesn’t appear in this list (or from cyclic or anticyclic permutations) it is 0), one solution for the generators is given by:

$$g_i = \frac{\lambda_i}{2}$$

and:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Explicitly compute the left and right hand sides of the algebra for $i = 7$, $j = 6$.

**Left hand side**: $[g_7, g_6] = [\frac{\lambda_7}{2}, \frac{\lambda_6}{2}] = \left[\begin{array}{ccc}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right] - \left[\begin{array}{ccc}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right] = \left[\begin{array}{ccc}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$.

**Right hand side**: $i f^{768} g_k = i f^{768} g_1 + i f^{768} g_2 + i f^{768} g_3 + i f^{768} g_4 + i f^{768} g_5 + i f^{768} g_6 + i f^{768} g_7 + i f^{768} g_8$.

$$f^{768} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad f^{768} = -\frac{\sqrt{3}}{2}$$

$$f^{376} = \frac{1}{2} \quad \Rightarrow \quad f^{376} = \frac{1}{2}$$

$$i f^{768} g_8 = \frac{i}{2} \frac{\lambda_8}{2} = -\frac{i}{2} \frac{\lambda_8}{2}$$

$$= i \frac{1}{2} \left(\begin{array}{cc}1 & 0 \\ -1 & -1\end{array}\right) \quad \text{BAM}$$