The weak interactions have many peculiar features that set them apart from QCD and ETH:

1. Every single matter particle in the SM exhibits weak interactions (each charged for ETH, only quarks for QCD)
2. The force mediators are massive (unlike photons and gluons)
3. The weak interactions violate parity, charge conjugation and CP
4. The weak interactions can change “flavor”, i.e. particle type ⇒ responsible for decays!

Perhaps the strongest part of the weak interactions is that they are not realized as a symmetry of the SM, at least not at room temperature type energies. We will eventually explain what this means and in fact this will solve the mediator mass issue by bringing in the Higgs mechanism. But more on that later.

To keep in line with our development of ETH and QCD as theories of local (or gauge) invariance, we will go ahead and formulate the weak interactions in terms of a gauge symmetry. This is relevant since at some point in the history of the universe this is how it appeared. More on that later!

Perhaps the most surprising feature of formulating the weak interactions in terms of a gauge symmetry is that to do so, we are forced to “unify” the weak force with ETH!
Electroweak Unification

It is often said that the total gauge symmetry group of the SM is $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_Y$.

However, this is not quite right. The correct groups are:

\[
\begin{align*}
\text{QCD} & \quad \text{Weak} \quad \text{Electroweak} \\
\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y & \quad \text{High Energy} \\
\quad \cup \quad & \quad \text{Chiral symmetry} \\
\text{SU}(3) \times \text{U}(1)_E & \quad \text{Low Energy}
\end{align*}
\]

So, we need to start with $\text{SU}(3)_c \times \text{U}(1)_Y$. The "L" in $\text{SU}(2)_L$ means that this symmetry is only relevant for "left-handed" fermion states. This is a bit of a misnomer since handedness has to do with helicity, whereas in actually the $\text{SU}(2)$ acts on states of definite chirality (which does match helicity for massless particles).
Recall: \( L_{\text{Dirac}} = (\gamma^\mu p_\mu + m) \bar{\psi} \psi \)

\[
\bar{\psi} = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\psi^t \ono^{L} \no^{R} \\
\psi^t \ono^{R} \no^{L}
\end{array} \right)
\]

To work with \( \gamma \)-component objects consider \( \psi^R = (\no^{R}) \) and \( \psi^L = (\no^{L}) \). Then \( \psi = (\no^{L} \no^{R}) = \psi^R + \psi^L \).

But recall that \( \gamma^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \gamma^L \gamma^R = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^L \gamma^R \), \( \gamma^L \gamma^R \).

However for \( \gamma = i \gamma^L \gamma^R \) \( \Rightarrow \overline{\psi} = (\gamma^L \gamma^R)^t \psi^R = i (\gamma^L \gamma^R)^t \psi^R = i \psi^R + i \psi^R = i \psi^R + i \psi^R = \overline{\psi} \).

Then we can write: \( L_{\text{Dirac}} = \left( \gamma^\mu \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) p_\mu \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) + m \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) \right) \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) \)

Note that: \( \overline{\psi} \gamma^\mu \psi = \overline{\psi} \gamma^\mu \psi = \overline{\psi} \gamma^\mu \psi = \overline{\psi} \gamma^\mu \psi = 0 \Rightarrow \text{All deriviative terms mixing L \& R vanish.} \)

\( \overline{\psi} \gamma^\mu \psi = 0 \Rightarrow \text{All mass terms adding L \& R vanish.} \)

\( L_{\text{Dirac}} = \left( \gamma^\mu \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) p_\mu \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) + m \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) \right) \left( \begin{array}{l}
\psi^R \\
\psi^L
\end{array} \right) \)

Everything here is in terms of \( \gamma \)-component objects.
\[ L_{\text{Dirac}} = (k \bar{c}) \bar{\Phi}_L \gamma^\mu \Phi_R + (k \bar{c}) \bar{\Phi}_L \gamma^\mu \Phi_R + nc \bar{\Phi}_L \gamma^\mu \Phi_R \]

Now that we have \( L + R \) in the game, we need a "doublet" for the \( SU(2)_L \) to act on.

Recall for quarks we introduced a triplet \( \Psi = \left( \begin{array}{c} u \\ d \end{array} \right) \) for \( SU(3) \).

We actually pair particles into left-handed doublets: \( \chi_L = \left( \begin{array}{c} \nu_e \\ \ell \end{array} \right)_L, \left( \begin{array}{c} \nu_\mu \\ \mu \end{array} \right)_L, \left( \begin{array}{c} \nu_\tau \\ \tau \end{array} \right)_L, \left( \begin{array}{c} l_1 \\ e \end{array} \right)_L, \left( \begin{array}{c} l_2 \\ \mu \end{array} \right)_L, \left( \begin{array}{c} l_3 \\ \tau \end{array} \right)_L \)

And have right-handed singlets: \( e_R, \mu_R, \tau_R, \nu_e, \nu_\mu, \nu_\tau, l_1, l_2, l_3 \)

What about \( \nu_e, \nu_\mu, \nu_\tau \)? They don't exist! At least not for the massive neutrino story.

For simplicity, we will focus on the first generation of leptons, i.e., \( \chi_L = \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L \) and ignore the mass term (since it never plays a role in generating interactions).

\[ L_{\text{Dirac}} = (k \bar{e}) \bar{\Phi}_L \gamma^\mu \Phi_R + (k \bar{e}) \bar{\Phi}_L \gamma^\mu \Phi_R \]
\[ L_{\text{Lagrangian}} = \mathcal{L}_e + \mathcal{L}_W + \mathcal{L}_H \]

This Lagrangian is invariant under gauge transformations, which are the symmetry operations that leave the physical laws unchanged.

The fields \( \gamma_5 \) and \( \gamma_5' \) are related to the gauge fields \( \phi \) and \( \phi' \) through the transformation rules.

Let's consider the transformation of the fields under the group.

\[ \mathcal{L}_e = -\frac{1}{4} F_{\mu} F^\mu + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \]

The effective potential \( V(\phi) \) is given by the difference in the vacuum energies.

The effective potential has a particularly interesting structure. It is a natural candidate for a Higgs mechanism.

Using the machinery of good old spin-1/2 from QM, we can think of this in terms of:

\[ \psi \rightarrow e^{i \theta} \psi \]

The fields \( W^+ \) and \( W^- \) change \( +1 \) and \( -1 \), respectively, under the group transformation.

But notice that \( W^+ \) changes to \( e^+ \) or \( e^- \) depending on whether the \( \pm 1 \) or \( -1 \) is applied, respectively.

The field \( \phi \) also changes to \( \phi' \) or \( \phi'' \) depending on whether the \( -1 \) or \( +1 \) is applied, respectively.

The \( \phi' \) field is neutral with respect to the electromagnetic force, whereas the \( \phi'' \) field is charged.
There are 2 big problems:

1) We know that the weak gauge bosons are massive, and we already know that the Peccei-Quinn term \( \delta \mu L / \lambda \) is not gauge invariant.

2) Recall that to have mass terms the squarks require both the L and R parts of \( \chi \) to combine, e.g., \( \Delta \chi^T \Delta \). However we have just considered a gauge theory where the L and R parts transform differently. There is no way we can expect \( \Delta \chi^T \Delta \) to be gauge invariant!

Both of these problems will be solved by the Higgs mechanism for mass generation. A crucial part of this process is the breaking of \( SU(2)_L \times SU(2)_R \to U(1)_E \).

We will have much more to say about how such a symmetry can be broken, but for now we will just highlight the implications for the electroweak interactions.

\( SU(2)_L \times SU(2)_R \) has 4 generators \( W^3, W^\pm, B \). After symmetry breaking to \( U(1)_E \) we only expect one symmetry generator to survive. Which one is it?

You might have thought it would be \( B \), then \( U(1)_E \times U(1)_E \), but that is not the case.

In actuality, the \( B \) "mixes" with the neutral \( \Delta \) from \( SU(2)_L \). We can form 2 orthogonal states:

\[ A_\mu = B_\mu \cos \theta + W^3_\mu \sin \theta \Rightarrow \text{The photon of } U(1)_E \]

\[ Z_\mu = -B_\mu \sin \theta + W^3_\mu \cos \theta \Rightarrow \text{The massive neutral } Z^0 \text{ boson of the weak interactions} \]

\[ \theta = \text{Wino mixing angle} \]

So it should be clear that we cannot identify just \( U(1)_E \) with the \( U(1)_E \) factor in \( SU(2)_L \times SU(2)_R \).

Going back to the original unified gauge group \( SU(2)_L \times SU(2)_R \), it wouldn't make much sense to call this a "unified" group if the \( SU(2)_L \) and \( U(1)_E \) factors had completely independent couplings \( g \) and \( g' \). It turns out that they are in fact related. However since we experience the broken version of this theory, it is more useful to know how the couplings to \( W^3, Z^0 \) and \( g \) are related.

Turns out: \( g \sin \theta = g' \cos \theta = g_x \) and \( g = g_x \frac{\sqrt{2}}{\sin \theta \cos \theta} \) while \( G_Z = \frac{g_x}{\sin \theta \cos \theta} \).