From here on out we will adopt the usual conventions used by particle physicists as opposed to relativity or string theorists or even formal field theorists.

Metric convention \[ g_{\mu\nu} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \]

With that in mind we should recall the change in the fields on us:

\[ V^\mu = \begin{pmatrix} \phi^0 \\ \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} \Rightarrow V'_\mu = \begin{pmatrix} \phi^0 - \phi^1 \phi^2 \\ \phi^1 - \phi^0 \phi^2 \\ \phi^2 - \phi^0 \phi^1 \\ \phi^3 \end{pmatrix} \Rightarrow V^\mu V'_\mu = \phi^0 \phi^1 - \phi^1 \phi^0 + \phi^2 \phi^3 - \phi^3 \phi^2 \]

\[ L_{\text{kin}} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \left( \frac{\lambda}{\kappa^2} \right) \phi^2 \Rightarrow \partial^\mu \phi^\mu + \left( \frac{\lambda}{\kappa^2} \right) \phi = 0 \]

\[ L_{\text{Lagr}} = \frac{1}{2} \left( \nabla^\mu \phi \nabla_\mu \phi - \kappa^2 \phi^2 \right) \Rightarrow \nabla^\mu \phi \nabla_\mu \phi + \frac{\kappa^2}{4} \phi^2 = 0 \]

\[ L_{\text{Diver}} = -\frac{1}{16} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \left( \frac{\lambda}{\kappa^2} \right) A_\mu A^\mu \Rightarrow \partial_\mu \left( \partial^\mu \phi - \phi \partial^\mu A^\mu \right) + \left( \frac{\lambda}{\kappa^2} \right) A^\mu = 0 \]
What would we like to calculate (and compare to experiment)?

Decays \( A \rightarrow \{ \beta + C, \, D + E, \, F + G + H \} \)

Decay channels indexed by \( i \):
\[
A \rightarrow \text{anything} \quad \Gamma_i = \sum_i \Gamma_i
\]

\( \Gamma \) are "decay rates" = probability per unit time of decaying into \( \beta \).

\[
\text{New:} \quad \frac{dN}{dt} = \Gamma_i \quad \text{for } \quad N(0) = N_0 \text{ e}^{-\Gamma_i t} \quad \rightarrow \quad T_{\text{avg}} = \frac{1}{\Gamma_i} = \text{"lifetime"}
\]

We will focus on calculating \( \sum_i \Gamma_i \) from which we can get \( \Gamma_{\text{tot}}, T_{\text{avg}} \).

The "likelihood" of a particular set of decay products is the scattering cross-section \( \sigma_i \). The total or inclusive cross-section for \( A+B \) is \( \sigma_{\text{tot}} = \sum_i \sigma_i \).

In contrast with a primitive scattering like firing an arrow at a target: \( A \rightarrow (0) \). In this simple case the likelihood is really determined by the actual cross-sectional area of the target.

In particle physics, scattering is much more complicated:
- Soft target (interaction w/ potential)
- Depends on identity of arrow
- Multiple ways to successfully "hit"
- Velocity dependent
- Primitive case the final state is "hit" or "no hit", whereas in particles there are many possible outcomes.

\[
\text{We will focus on calculating } \sigma_i.
\]

Sometimes our view is limited to a small slice of solid angle \( d\Omega \) (where detector sits), so we might instead need \( \sigma(\theta) \) which is typically only \( \theta \)-dependent.

\[
\theta
\]
Our interest is in relativistic quantum mechanical calculations of \( \Gamma_i, \delta_i \). This would really entail full QFT, but we will study and try to make sense of the results.

In both decays and scattering, the "likelihood" of an event is controlled by:

a) Kinematics (phase-space freedom), e.g., the larger the mass difference between in and out states, the more excess kinetic energy is liberated and this can be distributed in more ways in phase-space resulting in higher likelihood.

b) Dynamics (interactions), e.g., relative likelihoods governed by force strengths, intermediate states, etc.

These two influences actually quasi-separate in the final expressions for \( \Gamma_i \) and \( \delta_i \), so we can really handle them separately.

The kinematic contribution to \( \Gamma_i, \delta_i \) is summed up in Fermi's Golden Rule (which works for any interaction):

Decay: \( m_i \rightarrow m_1 + m_2 + \ldots + m_n \) (channel i)

\[
\Gamma_i = \frac{\alpha}{2\hbar c} \sum \frac{1}{M_i^2} \left( \begin{pmatrix} 1 \end{pmatrix} \delta \left( p_i - p_1 - p_2 - \ldots - p_n \right) \prod_{j \neq i} 2\pi \delta \left( p_j - \frac{q_j}{m_j} \right) \Theta \left( p_i \right) \right) \frac{d^3 p_i}{(2\pi \hbar)^3}
\]

Forces outgoing particles to have \( E > 0 \).

Forces outgoing particles to be real, i.e., satisfy mass-shell conditions.

Channels is constrained by \( 4\text{-momentum} \) conservation.

Enforces overall 4-momentum conservation.

\[
\delta_i = \frac{\alpha}{16\pi \hbar^2 c^4} \sum 1 \left( \begin{pmatrix} 1 \end{pmatrix} \delta \left( p_i - p_1 - p_2 - \ldots - p_n \right) \prod_{j \neq i} 2\pi \delta \left( p_j - \frac{q_j}{m_j} \right) \Theta \left( p_i \right) \right) \frac{d^3 p_i}{(2\pi \hbar)^3}
\]

Note: \( \delta_i = \frac{1}{S_1 \ldots S_n} \) where \( S_i = \# \) of identical out particles of type i

\( M_i^2 \) will carry all of the dynamical information.

The Golden Rule simply says that (dynamics aside) all kinematic configurations consistent with 4-momentum conserving, positive energy, and mass-shell conditions are equally likely. So the more of them there are, the higher the likelihood!!
At this point we usually can't go further since \(1\mu_0\) will often depend on \(p\), and so we need it before integrating. But in a few special cases the kinematics is so tightly constrained that we can go a bit further.

First, we can always break up \(d^6p = d^3p_0^\perp d^3p_0^\parallel\) and use \(S(p_0^\perp - h_0^\perp c^2) = S(p_0^\perp - p_0^\parallel - h_0^\perp c^2)\) to perform the \(d^3p_0^\parallel\) integral using the properties that:

\[
S(x^\parallel - h_x^\parallel c^2) = \frac{1}{ik} \int S(x - h_0^\perp + S(x + h_0^\perp) I 1c0, \text{constant}
\]

Then:

\[
\Gamma^2 = \frac{S}{2 \kappa m_0^{\perp}} \left( 1 + \frac{1}{x_0^{\perp}} \right) \frac{1}{r^2} \frac{1}{x_0^{\perp} + h_0^\perp c^2} \frac{3}{(2\pi)^2}
\]

Now for 2 cases that are so tightly constrained by the kinematics that we have enough \(S\)-functions in EGR to let us evaluate all of the integrals without the functional form of \(1\mu_0^\perp\).

1. **1-body decay** \(1 \rightarrow 2+3\): 
   \[
   \Gamma = \frac{S}{8 \kappa n_0^{\perp}} \text{MeV}^2 \text{where} \quad (\beta_1^2 = \frac{m_0^{\perp}}{\sqrt{h_0^{\perp} + h_0^{\perp} c^2}} - h_0^{\perp} c^2 = \frac{3\hbar^2}{\kappa})
   \]
   (at rest!)

   - Magnitude of momentum of either outgoing particle
   - Same since \(\beta_{out} = 0\)

   If you fix \(n_0\) and plot \(\Gamma\) as a function of \(n_0\), you will find that it grows with increasing mass difference.

2. **1-body scattering in CM frame** 
   \[
   \frac{d\sigma}{d\Omega} = \frac{(\kappa c)}{8\pi} S(\mu^2) \frac{\beta_{out}^2}{(E_1 + E_2)^2 - 1^2} \frac{d\Omega}{d\Omega} \quad 12 \rightarrow 3+4
   \]
   - Magnitude of momentum for either incoming particle
   - \(\beta_{out} = 0 = \beta_{out}\)