Our interest is in relativistic, quantum mechanical calculations of $\Gamma_i; \delta_i$. This would really entail full QFT, but we will study and try to make sense of the results.

In both decays and scattering, the “likelihood” of an event is controlled by:

a) Kinematics (phase-space freedom), e.g. the larger the mass difference between initial and final states, the more excess kinetic energy is liberated and this can be distributed in more ways in phase-space resulting in higher likelihood.

b) Dynamics (interactions), e.g. relative likelihoods governed by force strengths, intermediate states, etc.

These two influences actually quasi-separate in the final expressions for $\Gamma_i$ and $\delta_i$, so we can really handle them separately.

The kinetic contribution to $\Gamma_i, \delta_i$ is summed up in Fermi’s Golden Rule (which works for any interaction):

\[ \Gamma_i = \frac{\pi}{2k_{\pi}} \int \frac{d^4p_1 \cdots d^4p_n}{q^2 (p_1 \cdots p_n)} \left( \frac{2m_i}{q^2} \right)^{\frac{d+1}{2}} \delta(q - p_1 - \cdots - p_n) \delta(q - \sum p_i) \Theta(q^0) \delta^4(p_i - \frac{q}{2}) \]

\[ \delta_i = \frac{\pi}{2k_{\pi}} \int \frac{d^4p_1 \cdots d^4p_n}{q^2 (p_1 \cdots p_n)} \left( \frac{2m_i}{q^2} \right)^{\frac{d+1}{2}} \delta(q - p_1 - \cdots - p_n) \delta(q - \sum p_i) \Theta(q^0) \delta^4(p_i - \frac{q}{2}) \]

Note: $S = \frac{1}{2}, \frac{3}{2} \ldots$ where $S_i$ is # of identical outgoing particles of type $i$.

The Golden Rule simply says that (dynamics aside) all kinematic configurations consistent with 4-momentum conservation, positive energy, and mass-shell conditions are equally likely. So the more of them there are, the higher the likelihood!!
At this point we usually can’t go further since \( \Pi \) will often depend on \( \vec{p} \), and so we need \( \vec{p} \) before integrating. But in a few special cases the kinematics is so tightly constrained that we can go a bit further.

First, we can always break up \( d\Omega = d\Omega_1 d\Omega_2 \) and use \( \delta(\vec{p}_2^+ - \vec{p}_2^- - \vec{p}_3^- - \vec{p}_3^+) = \delta(\vec{p}_2^+ - \vec{p}_2^- - \vec{p}_3^- - \vec{p}_3^+) \) to perform the \( d\Omega \) integral using the properties that:

\[
\delta(x^+ - x^-) = \frac{1}{\pi} \ln 2 \int \delta(x^+ + \delta x + x^-) \, dx
\]

Then:

\[
\Pi_1 = \frac{s}{\sqrt{s}} \int \frac{d\Omega_1}{\sin^2 \theta_1} \delta^2(\vec{p} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \frac{1}{2 \sqrt{\vec{p}_1^2 + m_1^2}} \frac{1}{2 \sqrt{\vec{p}_2^2 + m_2^2}}
\]

\[
\Pi_2 = \frac{s}{\sqrt{s}} \int \frac{d\Omega_2}{\sin^2 \theta_2} \delta^2(\vec{p} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \frac{1}{2 \sqrt{\vec{p}_3^2 + m_3^2}} \frac{1}{2 \sqrt{\vec{p}_2^2 + m_2^2}}
\]

Now for 2 cases that are so tightly constrained by the kinematics that we have enough S-factors in \( \Sigma \) to let us evaluate all of the integrals without the functional form of \( \Pi \).

"2-body" decay \( 1 \to 2+3 \) \( \Pi = \frac{s \vec{p}_1 \vec{p}_2}{8 \pi m_1 c} \) \( \Pi \) \( \vec{p}_1, \vec{p}_2, \vec{p}_3 \) \( \vec{p}_1 = \vec{p}_2 = \vec{p}_3 \)

"2-body" scattering in \( \Sigma \) frame \( \delta \Sigma = \frac{s \vec{p}_1 \vec{p}_2}{(E_1 + E_2)^2} \) \( \vec{p}_1 \)

\( \vec{p}_1 = \vec{p}_2 = \vec{p}_3 \)
The Feynman Rules (or how to calculate $M$)

A very simple “toy” model:

$$\mathcal{L} = \frac{i}{4} \bar{\psi}_A \gamma^\mu \psi_A \bar{\psi}_B \gamma^\mu \psi_B - \frac{i}{4} \bar{\psi}_C \gamma^\mu \psi_C \bar{\psi}_D \gamma^\mu \psi_D + \frac{1}{2} \bar{\psi}_A \gamma^\mu \gamma^\nu \gamma^\rho \psi_A \bar{\psi}_B \gamma^\mu \gamma^\nu \gamma^\rho \psi_B + \frac{1}{2} \bar{\psi}_C \gamma^\mu \gamma^\nu \gamma^\rho \psi_C \bar{\psi}_D \gamma^\mu \gamma^\nu \gamma^\rho \psi_D - g \bar{\psi}_A \gamma^\mu \gamma^\nu \gamma^\rho \psi_A \bar{\psi}_B \gamma^\mu \gamma^\nu \gamma^\rho \psi_B$$

1. 3 real spin-0 particles $A, B, C$
2. They are their own antiparticles, e.g., $A \rightarrow \overline{A}$ or $A \rightarrow \gamma \rightarrow \text{no interaction}$
3. $\overline{A} \rightarrow \overline{B} + \overline{C}$
4. A basic interaction vertex

The first step is to draw the diagrams. Beginning with the initial and final states, e.g., we then “connect” them in all possible ways using the interaction vertices. Note that we can rotate the vertices, e.g., $\frac{\overline{A} \rightarrow \overline{B}}{B \rightarrow C}$, but we cannot change the particle content, e.g., $\frac{\overline{A} \rightarrow \overline{B}, C}{B \rightarrow C}$ is not allowed.

As first is might seem obvious, how to connect the pieces, but we quickly realize there are many complicated ways.

Example: “Decay of $A$ into $B + C$”

$$\begin{align*}
\text{Tree-level} & \quad \text{Vertex-correction} \quad \text{Self-energy-correction} \\
\text{1st order} & \quad \text{3rd order}
\end{align*}$$

We evaluate each diagram to get $M_i$ using these rules (will change somewhat for full $S_M$):

1. Label all momenta. $p_i$: external, $q_i$: internal with arrows next to lines. This lets us keep track of momentum flow (different from particle identity flow). For $p_i$, the arrows must go forward in time, but for $q_i$, it doesn’t matter.
2. For each vertex write a factor of $-ig$ (g is the coupling strength).
3. For each internal line we write a factor $\frac{1}{\sqrt{2m_i}}$. Note: $q_i = p_i$. Virtual!
4. For each vertex conserve 4-momentum $W(\sum p_i, n\rightarrow n; P_{\text{final}}, P_{\text{initial}})$ where $P = \sum p_i, q_j$.
5. Integrate everything you have written over all internal 4-momentum $W$ “factors” $\int \frac{d^n q_j}{(2\pi)^n}$.
6. After this you will have an overall $(3n)^5$ (proton-proton). Erase this and multiply by $i$ to get $M_i$.

As a trivial example we will evaluate the tree-level diagram above one step at a time:

1. $-ig$
2. $1$. strip internal lines
3. $-ig$
4. $-ig \left(2\pi\right)^5 \delta^4(p_i - p_k - p_j)$
5. strip internal lines
6. $-ig \left(2\pi\right)^5 \delta^4(p_i - p_k - p_j) \times i = g$