Mr. Feynman and Mr. Lagrangian

Recall the A\(\bar{A}\)C's:
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \phi^\dagger \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi^\dagger + \frac{1}{2} m^2 \phi \phi^\dagger + \frac{1}{2} m^2 \phi^\dagger \phi - \frac{1}{4} g \phi \phi \phi^\dagger \phi^\dagger - g \phi^\dagger \phi^\dagger \phi^\dagger \phi
\]

In evaluating diagrams built from \(-\frac{i}{c} Y^3\) we used: \(-ig\) vertex factor

\(\frac{1}{q^2-m^2}\) virtual particle propagators

There is actually a systematic way to extract the Feynman rules for a given Lagrangian, but to derive it requires QFT.
We will simply state the results.

**ABC example**

**Vertex Factors:** Write fields in momentum space \( i \gamma_\mu \partial_\mu \to p_\mu \)
\(\phi \to \frac{1}{c}\)

Erase the field variables

\(-ig\)

**Propagators:** Write the relevant free-particle each in momentum space \(\phi^\dagger \phi \phi \phi^\dagger \to \frac{(2\pi)^4}{\mathbf{p}^2 - m^2} \phi \phi^\dagger \to 0 \Rightarrow [\mathbf{p}^2 - (mc)^2] \phi = 0\)

and lose overall factor of \(\frac{1}{c}\)

Multiply the inverse of the term in brackets \(\frac{1}{\mathbf{p}^2 - m^2 c^2}\)
Consider the QED Lagrangian:
\[ L = i e A^\mu \partial_\mu \bar{\psi} \psi - \eta^\mu \phi \bar{\phi} - \frac{g}{16\pi} (\partial^\mu \phi - \partial^\nu \phi^* \partial_\nu \phi) A_\mu - \frac{1}{16\pi} (\partial^\mu \phi^* - \partial^\nu \phi^* \partial_\nu \phi) A_\mu A^\nu \]

The fundamental vertex is \( e^{-e} \bar{\psi} \psi \) or any other charged particle due to unitarity by \( A_\mu \).

The vertex factor we get from \( i \partial_\mu \bar{\psi} \psi A_\mu \) removing \( \sqrt{-i} \), \( \sqrt{i} \eta A^\mu \) giving:
\[ \pm \frac{g}{16\pi} (\bar{\phi} \gamma^\mu \phi - \bar{\phi} \gamma^\mu \phi^* \partial_\mu \phi) \]

For the propagators we can have both virtual photons and electron/positrons.

Electrons/Positrons use:
\[ i \partial^\mu \bar{\psi} \gamma^\mu \psi = 0 \Rightarrow \left( \gamma^\mu p_\mu - m c \right) \psi = 0 \Rightarrow \frac{p^\mu + m c}{p^2 - m^2 c^2} \]

For photons: Start with the massive Proca equation:
\[ \partial_\mu \left( \partial^\mu A^\nu - \partial^\mu A^\nu \right) + \left( \frac{e^2}{c^2} \right) A^\nu = 0 \]

\[ \partial_\mu \bar{\psi} \gamma^\mu \psi = 0 \]

We want this as one thing operating on \( A^\nu \) so let's massage it a bit. First get rid of \( \bar{\psi} \) and \( \psi \):
\[ p^\mu \bar{\psi} \gamma^\mu \psi + \bar{\psi} p^\mu \gamma^\mu \psi = 0 \]

Multiply by \( \gamma^\mu \) on both sides.
\[ (\gamma^\mu) \left( - p^\mu A^\nu + \bar{\psi} p^\mu \gamma^\mu A^\nu + (mc)^2 \right) \psi = 0 \]

\[ (\gamma^\mu) \left( - p^\mu + (mc)^2 \right) A^\nu + \bar{\psi} p^\mu A^\nu = 0 \]

\[ \left[ \gamma^\mu, - p^\mu + (nc)^2 \right] A^\nu = 0 \]

Everywhere replace \( \gamma \rightarrow \lambda \) Useful for weak interactions later

\[ \left[ \gamma^\mu, - p^\mu + (nc)^2 \right] A^\nu = 0 \]

For \( h \neq 0 \) the propagator is:
\[ \left[ \gamma^\mu, - p^\mu + (nc)^2 \right] A^\nu = \left[ \gamma^\mu, - p^\mu + (nc)^2 \right] \]

For \( h = 0 \) we also need transversality, i.e., \( \Delta^\mu A^\nu = 0 \) leaving
\[ \left[ \gamma^\mu, p^\mu \right] A^\nu = \left[ \gamma^\mu, p^\mu \right] = -\frac{\gamma^\mu p^\mu - p^\mu \gamma^\mu}{p^2} \]

\[ \text{Photon propagator} \]
QED \ (for \ new \ jazz \ colored \ leptons \ e, \ \mu, \ \tau \ and \ photons \ \gamma)\)

Diagrams are built from: \[
\begin{array}{c}
\text{e}^+ \\
\text{e}^-
\end{array}
\] (or \(\mu, \tau\) versions)

In the ABC theory, all else was simpler so: a) order was unimportant
\[
\text{b) we were guaranteed } \psi \text{ would be a scalar}
\]

In QED, e, \(\mu, \tau\) are spinors and \(\gamma\) is a vector so: c) order is important
\[
\text{b) we have to be careful to ensure } \psi \text{ is a scalar}
\]

One new complication is that in most experiments we used unpolarized in-states and sum over all out states.

This means in calculating \(\mathcal{M}\) we must a) average over incoming spin states
\[
\text{b) sum over outgoing spin states}
\]

Unitary matrices: \[
\begin{align*}
\text{Electron} & \quad \text{Positron} & \quad \text{Photon} \\
\hat{u} & = u^\gamma & \bar{u} & = u^\gamma & \hat{v} & = v^\gamma & \hat{e} & = \hat{e}^\gamma \\
\hat{u} \, (\gamma \mu - m) & = 0 & \bar{v} & = v^\gamma & \bar{u} \, (\gamma \mu + m) & = 0 & \hat{e} & = \hat{e}^\gamma \\
\hat{u} \, (\gamma \tau - m) & = 0 & \bar{v} & = v^\gamma & \bar{u} \, (\gamma \tau + m) & = 0 & \hat{e} & = \hat{e}^\gamma
\end{align*}
\]

Orthonormality: \[
\begin{align*}
\hat{u} (\gamma \tau - m) & = 2mc \, \delta_{a'}^b \\
\bar{v} & = v^\gamma \\
\hat{e} & = \hat{e}^\gamma
\end{align*}
\]

Completeness: \[
\sum_a \hat{u} (\gamma \tau - m) \bar{v} (\gamma \mu + m) = \sum_a \hat{u} (\gamma \tau - m) \bar{u} (\gamma \tau + m) = 1
\]

The Dirac spinors for \(\beta \neq 0\) in the new convention are:

\[
\begin{align*}
\hat{u}^{[1]} & = \sqrt{\frac{E + m_c^2}{C}} \\
\hat{u}^{[2]} & = \sqrt{\frac{E + m_c^2}{C}} \\
\bar{v}^{[1]} & = \frac{c \beta_v}{E + m_c^2} \\
\bar{v}^{[2]} & = \frac{-c \beta_v}{E + m_c^2}
\end{align*}
\]

And for the spinor matrices:

\[
\begin{align*}
\hat{Y} & = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \hat{\gamma} & = \begin{pmatrix} 0 & \beta_i \beta_j \\ \beta_i \beta_j & 0 \end{pmatrix} & \hat{\gamma} & \hat{\gamma}
\end{align*}
\]

\[
\begin{align*}
\hat{u} & = \begin{pmatrix} \frac{c \beta_v}{E + m_c^2} \\
\frac{-c \beta_v}{E + m_c^2} \end{pmatrix} & \bar{v} & = -\sqrt{\frac{E + m_c^2}{C}} \\
\hat{u} & = \begin{pmatrix} \frac{c \beta_v}{E + m_c^2} \\
\frac{-c \beta_v}{E + m_c^2} \end{pmatrix} & \bar{v} & = -\sqrt{\frac{E + m_c^2}{C}}
\end{align*}
\]

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Feynman Rules for QED

1. Draw diagram and label matter lines w/ arrows to distinguish particles/anti-particles, e.g. 

\[ \begin{array}{c} \text{electron} \rightarrow p^- \rightarrow p^+ \\ \text{positron} \rightarrow p^+ \rightarrow p^- \end{array} \]

Label momenta: external \[ p_1, p_2 \] (along \( l \))

2. Each internal line gets a factor according to:

\[ \begin{array}{c} \text{Number factors w/ momenta, e.g.} \quad \rightarrow p \rightarrow \Rightarrow 1 \times i \end{array} \]

\[ \text{Note: Each factor is a 4-vector or \textit{4-tensor}.} \]

3. Each vertex gets a factor \( ig_\mu \gamma^\mu (g_\mu = e\sqrt{\frac{\hbar}{2m}}) \) \( \text{Note: Overall spin matrix.} \)

4. Each internal line gets a factor:

\[ \text{Electrons/Positrons: } \frac{i(\gamma^\mu p^\mu + m^\mu)}{q^2 - m^2 c^2} \]

\[ \text{Overall spin matrix.} \]

\[ \text{Photons: } \frac{i\gamma^\mu}{q^2} \]

\[ \text{Overall tensor.} \] (\( \mu, \nu \) match vertex indices)

5. Conserving Momentum at each vertex w/ \( (2\pi)^4 \delta^4 \left( p_{\text{in}} + q_{\text{in}} - p_{\text{out}} - q_{\text{out}} \right) \)

6. Integrate over internal momenta w/ \( \int \frac{d^4q}{(2\pi)^4} \) for each \( q \).

7. Cancel overall \( (2\pi)^4 \delta^4 \left( p_{\text{in}} - p_{\text{out}} \right) \) and \( \times i \) to get \( H \).

8. Antisymmetrize between diagrams related by switching 2 incoming electrons/positrons, 2 outgoing electrons/positrons or one incoming electron/positron with one outgoing positron/electron.

9. To get the order right for spinor elements (since we suppress indices) make “spinor sandwiches from matter lines” by starting w/ an outgoing matter particle line (\( \bar{u} \)) or incoming antimatter line (\( \bar{u} \)) and following along only matter segments writing vertex factors and internal matter propagators as we encounter them, until we eventually emerge on an outgoing antimatter (\( \bar{u} \)) or incoming matter (\( u \)) line. The photon factors are less tricky since index notation gets them right.
Example:

\[
e + e \rightarrow e + e
\]

Consider \( e^+ + e \rightarrow e^+ + e \):

1. \( P_1 \) 2. \( P_3 \)

\[ M_1 \]

\[ M_2 \]

switch one outgoing \( e^+ \) with one incoming \( e^+ \).

\[ h = h_1 + h_2 \]

\[ \Rightarrow h = h_1 + h_2 \]

\[ S_0 \]

So, \( h = h_1 - h_2 \)