Feynman Rules for QED

1. Draw diagram and label matter lines w/ arrows to distinguish particles/anti-particles, e.g., electron → \( p_e \), positron \( \bar{p}_e \)

   Label momenta: a) external \( \rightarrow \) \( \bar{p}_e \) → \( p_e \) (along \( \mathbf{P} \))

   b) internal \( \rightarrow \) \( \mathbf{p}_i \) \( \mathbf{p}_f \) \( \mathbf{q} \)

2. Each **external** line gets a factor according to:

   Number factors w/ momenta, e.g., \( \rightarrow \) \( \Rightarrow \) \( U(c) \)

   Note: Each factor is a \( \psi \)spinor or \( \chi \)vector.

3. Each vertex gets a factor \( i g \sigma^a \frac{q^a}{2} \left( \gamma^\mu \right) \) Note: Overall spin matrix.

4. Each **internal** line gets a factor:

   - Electrons/Positrons \( \frac{i(g_{\mu\nu} + \epsilon_{\mu\nu} \gamma^5)}{2E_e} \) Overall spin matrix.
   - Photons \( \frac{-i g_{\mu\nu}}{g_s} \) Overall tensor

   (both match vertex indices)

5. Conserved momentum at each vertex \( \Rightarrow \) \( \delta q^\mu \left( p_i - p_f - q + q_{\text{out}} - q_{\text{in}} \right) \)

6. Integrate over internal momenta \( \Rightarrow \) \( \int \frac{d^4 q}{(2\pi)^4} \) for each \( q \).

7. Cancel overall \( \frac{-1}{4} g_{\mu\nu} \left( p_i - p_f - q + q_{\text{out}} - q_{\text{in}} \right) \) and \( x_i \) to get \( \chi \).

8. Antisymmetrize between diagrams related by switching 2 incoming electrons/photons, 2 outgoing electrons/photons or one incoming electron/positron with one outgoing positron/electron.

9. To get the order right for spinor elements (since we suppress indices) make “spinor sandwiches” from matter lines’ by starting w/ an outgoing matter particle and tracing it back along pure matter lines writing polarization and vertex factors as needed. The photon factors are less tricky since index notation gets them right.
Example:

\[ e + e \rightarrow e + e \]

\[ \sum \left( \bar{\psi} \gamma^\mu u(1) \bar{\nu}(4) \right) \left( \bar{\lambda} \gamma^\nu \bar{u}(2) \lambda(5) \right) \left( \bar{\psi} \gamma^\nu \bar{u}(2) \lambda(5) \right) \left( \bar{\lambda} \gamma^\mu u(1) \bar{\nu}(4) \right) \]

\[ \Rightarrow h = h_1 \]

\[ B_7 \text{ step 8: } M = h_1 - h_2 \]

Consider \( c \downarrow e \rightarrow e + e \):

1. 

\[ M_1 \]

\[ h = h_1 + h_2 \]

2. 

\[ M_2 \]

\[ \text{Switch one outgoing } e \text{ with one incoming } e^\dagger. \]

\[ h = h_1 - h_2 \]

\[ S_0 \text{ this is the same as in case 1.} \]
Consider $n \rightarrow \infty$

$$\Rightarrow \mathcal{H} = \sum (a) \gamma^a \mathcal{U}^{(a)}(\mathcal{P}) \gamma^a \mathcal{U}^{(a)}(\mathcal{P}^{-1}) \gamma^a \mathcal{U}^{(a)}(\mathcal{P})$$

Once we have $\mathcal{H}$, we can evaluate it given the following:

$$\mathcal{U}(i) = \mathcal{U}^{(a)}(\mathcal{P}) + \mathcal{U}^{(a)}(\mathcal{P}^{-1}) \gamma^a \mathcal{U}^{(a)}(\mathcal{P})$$

However, in most experiments we average over $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and sum over $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$.

To do the avg/sun we first note that $\mathcal{H}$ contains terms of the form:

- Some combination of spin matrices
- Label to distinguish from other ones
- Wronskian sandwich

Note: We give them a different label. In particular if $\gamma^a$ carries spacetime indices $\mu, \nu$ then $\gamma^b$ will need different indices $\alpha, \beta$ to avoid conflicting contractions.

Then:

$$\mathcal{M} = \sum \mathcal{U}(a) \mathcal{P}_a \mathcal{U}(b) \left[ \mathcal{U}(a) \mathcal{P}_a \mathcal{U}(b) \right]^* \cdots$$

This entry is a number in spin space so we can freely transpose it to form $^t$.

$$= \cdots \mathcal{U}(a) \mathcal{P}_a \mathcal{U}(b) \left[ \mathcal{U}(a) \mathcal{P}_a \mathcal{U}(b) \right]^{+}$$

$$= \mathcal{U}(b)^{+} \mathcal{P}_a \mathcal{U}(b) = \mathcal{U}(b)^{+} \mathcal{P}_a \mathcal{U}(a)$$

Now the first step will be to sum so we can use $\sum_{s} \mathcal{U}(s)^{+} \mathcal{U}(s) = 1$

$$\sum_{s} \mathcal{M}_s = \cdots \mathcal{U}(a) \mathcal{P}_a \left( \mathcal{P}_b + \gamma^c \mathcal{C} \right) \mathcal{P}_b \mathcal{U}(a) \cdots$$

Now we can use that $\mathcal{R} = \mathcal{R} = \mathcal{M} \mathcal{C}$

$$\sum_{s} \mathcal{M} = \cdots \sum_{s} \mathcal{P}_a \left( \mathcal{P}_b + \gamma^c \mathcal{C} \right) \mathcal{P}_b \mathcal{U}(a) \cdots$$

Now sum over $s_a$ to get:

$$\sum_{s_a, s_b} \mathcal{M} = \cdots \sum_{s_a, s_b} \mathcal{P}_a \left( \mathcal{P}_b + \gamma^c \mathcal{C} \right) \mathcal{P}_b \left( \mathcal{P}_b + \gamma^c \mathcal{C} \right) \cdots$$
If at some point we had summed over $\bar{u} \rightarrow \bar{\chi} - n_c$.

Note: There are no spinors left in the expression! Only $\bar{n}_\mu$'s and $\bar{\chi}$'s!

To impose sum over $\bar{\chi}, \bar{\psi}$ replace $\bar{\psi}(\bar{\chi}) u(x) [\bar{\psi}(\bar{\chi}) u(x)]^* \rightarrow \text{Tr} \left[ \gamma_5 (p_0 + n_c) \gamma_\mu (p_0 + n_c) \right]

Let's put this result to work:

\[
\langle \chi \bar{\chi} \rangle = \frac{9g^2}{(\alpha_s - \beta_0)} \text{Tr} \left[ \gamma_5 (p_0 + n_c) \gamma_\mu (p_0 + n_c) \right] \text{Tr} \left[ \gamma_5 (p_0 + n_c) \gamma_\mu (p_0 + n_c) \right] g_{\mu \nu} g_{3 \nu'}
\]