In our discussion of rotations in 3D we have encountered scalars, vectors, and tensors. There are 0, 1, 2 and higher dimensional representations of the rotation group.

What about a 3D representation of 3D rotations?

We would need 2x2 matrices satisfying $[q_{ij}, q_{kl}] = i e^{i q_{ij} q_{kl}}$ when $i,j,k,l = 1,2,3$ and $[q_{ij}, q_{kl}] = i q_{ij} q_{kl}$ in 3D.

These work: $q_{xy} = (\cos(\theta), i \sin(\theta))$ $q_{y,z} = (\cos(\theta), i \sin(\theta))$ $q_{x,y} = (\cos(\theta), i \sin(\theta))$

where $\theta, \phi, \lambda$ are the Pauli spin matrices.

Now we can build: $R_x(\theta) = e^{i q_{xy}}$ and similarly for $R_y$ and $R_z$.

Satisfy $U^\dagger U = I$ for $SU(2)$ which act on complex 2-component spinors $\chi$.

Often we write $x \mapsto \chi \mapsto \chi U = U\chi$.

Note: We will not use spin indices in this class, so we will rely on matrix manipulations.

So $SO(3) \sim SU(2)$, at least near the identity (which is all the Lie algebra knows about).

Globally, however there is a difference: $SO(3), \quad R_x(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim SU(2), \quad R_x(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}

This is the double cover of $SO(3)$.

of course $R_x(4\pi) = I$ for both!

There is a certain sense in which spinors and $SU(2)$ probe geometry more deeply than coordinates, scalars, vectors, $SO(3)$, etc.

By "probe more deeply" I mean they contain more information. Sometimes people say that spinors know about the square root of the geometry.

In fact, if we consider the anti-commutator of the Pauli matrices we find: $[\sigma_a, \sigma_b] = 2 \delta_{ab}$ for $a,b = 1,2,3$.

Example: $0 \sigma_3 + 0 \sigma_3 + 0 \sigma_3 = (0 0 0 0) + (0 0 0 0) + (0 0 0 0) = (0 0 0 0)$ as expected since $\mathbb{S}^2 = 0$.

$0 \sigma_3 + 0 \sigma_3 + 2 \sigma_3 = 2 (0 0 0 0) = 2 (0 0 0 0)$ as expected since $\mathbb{S}^2 = 1$

It might seem silly, but recall that $q_{ij} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is the metric of $\mathbb{R}^2$. This will come in handy later.

Another illustration of this is a lesson from QM: If we only have integer spin states at our disposal, 0, 1, 2, ... then by combining spins we can only ever build more integer spin states.

However, if we allow 1/2 integer spin states, then we can build 0 or whole integer states just like 1/2 spin states, e.g. $\frac{1}{2} - \frac{1}{2} = 0, \frac{1}{2} + \frac{1}{2} = 1$. 

Lecture8. Spinors I Page 1
To finish up, we need to determine how to build an invariant (for Lagrangians) out of spinors.

Following our usual recipe: If \( \chi \rightarrow \chi' = e^{-i \frac{\theta \cdot \sigma}{2}} \chi \) and \( \chi \rightarrow \widetilde{\chi} = (e^{i \frac{\theta \cdot \sigma}{2}})^\dagger \chi \), then \( \widetilde{\chi}' \) is invariant.

But recall how we form \( \widetilde{\chi} \) from \( \chi \): \( \widetilde{\chi} = (\sigma \chi) \) where \( (e^{i \frac{\theta \cdot \sigma}{2}})^\dagger g e^{i \frac{\theta \cdot \sigma}{2}} = g \).

However, for \( SU(2) \) we already know that \( U^\dagger U = 1 \) so \( g = I \) and we can say \( \widetilde{\chi} = (\sigma \chi) = \chi \) and then \( \chi^\dagger \chi \) is invariant!

**Note:** All of the 6 matrices are Hermitian, i.e. \( \sigma_i^\dagger = \sigma_i \), \( \theta \) is real so

\[
U^\dagger = (e^{i \frac{\theta \cdot \sigma}{2}})^\dagger = e^{-i \frac{\theta \cdot \sigma}{2}} = U^{-1} \quad \text{This will not be the case later!}
\]

You can see more explicitly by Taylor expanding

\[
\left[ I + \left( \frac{1}{2} \theta \cdot \sigma \right) + \frac{1}{2} \left( \frac{1}{2} \theta \cdot \sigma \right)^2 \sigma_i \right]^\dagger
\]

\[
= I + \left( -\frac{1}{2} \theta \cdot \sigma \right) + \frac{1}{2} \left( -\theta \cdot \sigma \right)^2 (\frac{1}{2} \theta \cdot \sigma) + \ldots
\]

\[
= I + \left( -\frac{1}{2} \theta \cdot \sigma \right) + \frac{1}{2} \left( -\theta \cdot \sigma \right)^2 (\frac{1}{2} \theta \cdot \sigma) = e^{-i \frac{\theta \cdot \sigma}{2}}
\]
Now it is time to repeat this procedure for special relativity.

The Lorentz transformations as they act on coordinates/vectors form \( SO(1,3) \) so let's explore its algebra.

We expect 6 generators corresponding to: \( R_x, R_y, R_z, B_x, B_y, B_z \).

We will call the corresponding generators: \( J_x, J_y, J_z, K_x, K_y, K_z \).

Fortunately we already know a lot about the \( J \)'s: \( J_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) \( J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) \( J_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) \( J_x \). From which we can also get \( SU(2) \):

\[
[J_i, J_j] = i \epsilon^{ijk} J_k
\]

If we take the various boosts and again consider their Taylor expansion, then using the exponential map \( \beta = \exp(i K_{ab}) \) we find:

\[
K_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[
K_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[
K_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Now is where it gets interesting. By brute force one can show:

\[
[K_i, K_j] = -i \epsilon^{ijk} J_k
\]

2 boosts \( \Rightarrow \) rotation

\[J_i, K_j \]

rotation \( \Rightarrow \) boost

Question: Can the boosts alone form a subgroup of \( SO(1,3) \)? No

What about rotations? Yup

So unfortunately the boosts and rotations of \( SO(1,3) \) do not cleanly split from each other.

But...
Let's play an old math/physics trick:

Define \( \overline{J}_+: = \frac{1}{2} \begin{pmatrix} J_i & iK_j \end{pmatrix} \) \( \Rightarrow \)

\[
\begin{align*}
[\overline{J}_+; \overline{J}_+] &= i e^{i\theta} \overline{J}_+ \Rightarrow SO(3) \\
[\overline{J}_-; \overline{J}_-] &= i e^{i\theta} \overline{J}_- \Rightarrow SO(3) \\
[\overline{J}_+; \overline{J}_-] &= 0 \Rightarrow \text{These } SO(3) \text{ don't mix.}
\end{align*}
\]

So we find that at least near the identity \( SO(1,3) \approx SO(3) \times SO(3) \).

Remember this is not a split into 3 boosts and 3 rotations!

Now everything so far has been in terms of coordinates (scalars, vectors, tensors, etc.), but we can immediately see how to introduce spinors.

We utilize \( SO(1,3) = SO(3) \times SO(3) \approx SU(1) \times SU(2) \).

Each of these will act on a complex 2 component object, so our total spinor in 4D has 4 complex components!

This is most unfortunate since now we have 4 component vectors and 4 component spinors, but the components mean totally different things. This is only a misfortune in 4D:

\[
\begin{array}{cccccccccc}
3D & 4D & 5D & 6D & 7D & 8D & 9D & 10D \\
\text{vector} & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{spinor} & 1 & 2 & 4 & 8 & 16 & 32 & \\
\end{array}
\]

The counting goes: For each independent plane you can define an independent \( \gamma \)-field or 2-component spinor going \( \frac{1}{2} \times \frac{1}{2} \) states depending on \( d \) even or odd.
We now need to determine how these 4-component spinors transform and then how to build an invariant.

You might think we could just use the 4x4 matrices we already have for \( \mathbf{K} \) and \( \mathbf{F} \), but remember these act on coordinate related quantities, not spinors.

So what should we use? There are numerous ways to get at the answer, but we will use the deepest based on the idea of the square root of the geometry.

Recall: \( \delta_{i \bar{j}} \delta_{j \bar{i}} = 2 \delta_{i \bar{i}} \Gamma_{i \bar{i} \bar{i}} \Rightarrow \chi \rightarrow \chi' = e^{\frac{i}{2} \bar{\sigma} \cdot \bar{\sigma}} \chi \)

The perhaps: \( \{ \chi^0, \chi^1, \chi^2, \chi^3 \} \rightarrow \chi^0, \chi^1, \chi^2, \chi^3 \phi \Rightarrow \chi \rightarrow \chi' = e^{\frac{i}{2} \bar{\sigma} \cdot \bar{\sigma}} \phi \)

Unfortunately this won't work! We expect \( \chi^1, \chi^2, \phi \), so this is only 4 distinct transformations, but we know there should be 6!

Fortunately the answer is hiding in our basic notation.

If instead of \( \delta_{i \bar{j}} \delta_{j \bar{i}} \) we think of \( \delta_{i \bar{j}} \delta_{j \bar{i}} = \mathbf{I} \), \( \mathbf{F} \) and \( \mathbf{K} \) (see (13))

Rotation around \( \chi \) is really in the \( y-z \) plane.

Then we can think of: \( \chi \rightarrow \chi' = e^{\frac{i}{2} \bar{\sigma} \cdot \bar{\sigma}} \chi \) (which is the same as the above).

If we call these \( \delta^{\mu \nu} = \{ \delta^{00}, \delta^{01}, \delta^{02}, \delta^{03}, \delta^{11}, \delta^{12}, \delta^{13}, \delta^{22}, \delta^{23}, \delta^{33}, \delta^{01}, \delta^{02}, \delta^{03}, \delta^{11}, \delta^{12}, \delta^{13}, \delta^{22}, \delta^{23}, \delta^{33} \} \)

The parameterizing the transformation with angles: \( \{ \alpha, \beta, \gamma, \Theta, \Phi, \Psi \} \equiv \{ \omega_0, \omega_1, \omega_2, \omega_3, \omega_3, \omega_3 \}

Then we can write our transformation: \( \chi \rightarrow \chi' = e^{\frac{i}{2} \bar{\sigma} \cdot \bar{\sigma}} \chi \) giving \( \chi \rightarrow \chi' = e^{\frac{i}{2} \bar{\sigma} \cdot \bar{\sigma}} \chi \) or \( \chi \rightarrow \chi' = e^{\frac{i}{2} \bar{\sigma} \cdot \bar{\sigma}} \chi \).
Without any further ado, I present (at least one set of) the Dirac $\gamma$ matrices:

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{w/} \quad \delta_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta_+ = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

Example: \[ \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

These have some nice properties:

- Recall \( \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu} \Gamma_{\mu\nu} \)
- Then \( (\gamma^0)^4 = -1 \), \( (\gamma^i)^4 = 1 \)
- And \( \gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu = 0 \) if \( \nu \neq \mu \) since \( \eta^{\mu\nu} \) is diagonal.

or \( \gamma^\nu \gamma^\mu = -\gamma^\mu \gamma^\nu \)

We can now explicitly form the generators:

\[ \sigma^i = -\frac{i}{4} [\gamma^0, \gamma^i] = -\frac{i}{4} \left[ \begin{array}{cc} \gamma^0 \gamma^i - \gamma^i \gamma^0 \end{array} \right] \]

\[ = -\frac{i}{4} \left[ \begin{array}{cc} 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{array} \right] \]

\[ = \frac{i}{4} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

Note: We now see why we needed the \( \frac{i}{4} \) in the definition. The transformation now reduces to the usual \( \text{SU}(2) \) transformation on each pair of spinor indices, i.e., \( \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \) \( \text{SU}(2) \)

\( \delta^\gamma \psi = \frac{i}{4} \epsilon^{\gamma \mu \nu} \begin{pmatrix} 0 & \delta_+ \\ \delta_- & 0 \end{pmatrix} \psi \) e.g., \( \delta^0 = \frac{i}{4} \begin{pmatrix} 0 & \delta_3 \\ \delta_3 & 0 \end{pmatrix} \)