You must answer all of the questions below. If on any problem you find yourself doing a ridiculous amount of tedious work, you are probably doing something wrong! Please use the metric convention from the early part of the course that $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. Challenge questions are not for credit!

1. Consider the group of transformations in 3D which carry the corners of an equilateral triangle into corners.
   a) Begin by drawing pictures of all possible configurations and identify the transformations that lead to each.
   b) Construct a faithful linear (matrix) representation of this group, showing the matrix form of all elements of the group. **Hint:** It helps to identify the "basic" transformations from which you can get everything else by repeated application.
   c) *(Optional challenge question)* Does this group form a subgroup of $SO(3)$?

\[ \begin{array}{c}
\text{a)} \quad \begin{array}{ccc}
A & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & R_{120} \\
C & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \\
B & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & 
\end{array} \\
\text{b)} \quad \begin{array}{ccc}
I & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & R_{120} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
R_{120}^{-1} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \end{array} \\
\text{c)} \quad \begin{array}{ccc}
F & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} & F R_{120} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
F R_{120}^{-1} & \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & = (F R_{120})^{-1} 
\end{array}
\end{array} \]

Even though our transformations in this case are real $3 \times 3$ matrices which are in fact orthogonal, i.e. $M^T M = I$, they do not all have $\det = +1$, e.g. $\det F = -1 = \det (F R_{120}) = \det (F R_{120}^{-1})$. So these cannot constitute a subgroup of $SO(3)$, even though they are obviously a subgroup of the rotations in 3D. However...
There are many other representations possible, e.g. a 6-D one, but perhaps the most straightforward is to use the actual 3D rotation matrices themselves!

Then \( T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

\[ R_{yz}(120) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(120) - \sin(120) \\ 0 & \sin(120) \cos(120) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \]

\[ R_{yz}(240) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \]

\[ T = R_{xy}(180) = \begin{pmatrix} \cos(180) & -\sin(180) & 0 \\ \sin(180) & \cos(180) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ R_{xy}(180) R_{yz}(120) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad R_{xy}(180) R_{yz}(240) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \]

In contrast to the previous representation, this time our matrices are obviously elements of SO(3), in particular they all have det = +1.

You can check that they satisfy the definitions of a group, but the main criterion is to verify closure.
2. If \( A^\mu = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) and \( B^\nu = \begin{pmatrix} e \\ f \end{pmatrix} \) are vectors in \( \mathbb{R}^3 \), consider \( C^{\mu \nu} = A^\mu B^\nu = \begin{pmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{pmatrix} \). The pattern of the multiplication to get the components of \( C^{\mu \nu} \) should be obvious.

a) What are the components of \( C^{\mu \nu} \)?

b) Explicitly confirm that \( C^{\mu \nu} = A^\mu B^\nu \) under a rotation by \( \theta = 90^\circ \) in the \( y - z \) plane. This means you should transform \( C^{\mu \nu} \) as a second rank tensor, then transform each of \( A^\mu \) and \( B^\nu \) as vectors, take their product (as done above) and show that the results agree.

c) (Optional challenge question!) Does this mean that we can write any tensor of the form \( M^{\mu \nu} \) as the product of two dual vectors? Explain why or why not.

---

2a) Since we are working in \( \mathbb{R}^3 \) the metric is trivial, i.e. \( g^{\mu \nu} = \mathbb{I}^{\mu \nu} \).

So raising or lowering indices does not change anything, \( C^{\mu \nu} = \mathbb{g}^{\mu \nu} \mathbb{g}^{\nu \lambda} C_{\lambda \nu} = \mathbb{I}^{\mu \nu} \mathbb{I}^{\nu \lambda} C_{\lambda \nu} \Rightarrow C^{\mu \nu} = \begin{pmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{pmatrix} \).

2b) \( A^\lambda \rightarrow A^\lambda' = \mathbf{\Lambda}_{\lambda \mu} A^\mu = \begin{pmatrix} a \\ c \\ -b \end{pmatrix} \) \( B^\nu \rightarrow B^\nu' = \mathbf{\Lambda}_{\nu \mu} B^\mu = \begin{pmatrix} e \\ f \end{pmatrix} \)

Then \( A^\lambda' B^\nu' = \begin{pmatrix} cd & af - ae \\ cd & cf - ce \\ -bd & be - bf \end{pmatrix} \)  

2c) No! \( M^{\mu \nu} \) generally has \( N^2 \) independent elements, while \( A^\lambda A^\nu \) has \( 2N \).
3. The generators of SU(3) can be written as $g_i = \frac{\lambda_i}{2}$ where:

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
$$

where the nonzero associated structure constants of the Lie Algebra are

$$f^{123} = 1, \quad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

with $f^{ijk}$ are totally antisymmetric in the three indices, i.e. $f^{ijk} = -f^{jik}$.

a) Do any pair of these generators commute? If so, identify at least one pair. If not argue why.

b) *(Optional challenge question)* Explain how you could go about finding a different basis of generators for this group.

\[ a) \text{Any pair } i,j \text{ for which } f^{ijk} \text{ (or any permutation) is zero will commute by virtue of the Lie algebra. So for example } \lambda_1 \text{ and } \lambda_8 \text{ commute. Other pairs: } (\lambda_2, \lambda_8) \text{ and } (\lambda_3, \lambda_8) \]

\[ b) \text{To get a different basis, we could either re-solve the Lie algebra expression, or just do an SU(3) transformation of the basis above (bearing in mind that each } \lambda \text{ has 2 color indices, i.e. } \lambda_{ij}, \text{ so each color index would be transformed).} \]
4. Recall that for 4-vectors the dot product is $P^1 \cdot P^2 \equiv P^1_{\mu} P^2_{\mu}$.
   a) Evaluate $Tr(y^\mu y^\nu y^\lambda y^\rho) P^1_{\mu} P^2_{\nu} P^3_{\lambda} P^4_{\rho}$ in terms of 4-vector dot products, e.g. $P^1 \cdot P^2$, etc.
   b) (Optional challenge question) Demonstrate that your answer to part (a) is invariant under the usual permutation symmetry of the trace.

a) From Hw1 we know:

$$Tr(y^\mu y^\nu y^\lambda y^\rho) = \eta (\eta^\mu \eta^\nu \eta^\lambda \eta^\rho - \eta^\mu \eta^\nu \eta^\lambda \eta^\rho)$$

So this becomes:

$$\eta (\eta^\mu \eta^\nu \eta^\lambda \eta^\rho - \eta^\mu \eta^\nu \eta^\lambda \eta^\rho)
+ \eta^\mu \eta^\nu \eta^\lambda \eta^\rho
= \eta (\eta^\mu \eta^\nu \eta^\lambda \eta^\rho)$$

b) The trace is cyclic so $Tr(y^\mu y^\nu y^\lambda y^\rho) = Tr(y^\lambda y^\nu y^\mu y^\rho)$

Using the expression above we find:

$$Tr(y^\mu y^\nu y^\lambda y^\rho) = \eta (\eta^\mu \eta^\nu \eta^\lambda \eta^\rho - \eta^\mu \eta^\nu \eta^\lambda \eta^\rho)$$

Using the symmetry of the metric, i.e. $\eta^\mu \eta^\nu = \eta^\nu \eta^\mu$

this becomes:

$$= \eta (\eta^\mu \eta^\nu \eta^\lambda \eta^\rho - \eta^\mu \eta^\nu \eta^\lambda \eta^\rho)$$

which is the same as above.
5. a) Confirm that \( P_+ = \frac{1}{2} \left( 1 + \frac{\hbar}{\hbar} S^x_\parallel \right) \) is a projection operator, i.e. that \( P_+ P_+ \psi^{(1)} = P_+ \psi^{(1)} \). You may do this either in terms of an explicit example or by manipulation of the operator definition.

b) (Optional challenge question) Do the set of projection operators \( P_\pm \) along with the identity form a group? If so, what is its multiplication table?

a) Consider \( P_+ (P_+ 4^{(1)}) = \frac{1}{4} (1 + \frac{1}{\hbar} S^x) (P_+ 4^{(1)}) \)

You proved in HW5 that \( 4^{(1)} \) is an eigenstate of \( S^z_\parallel \) with eigenvalue \( \pm \hbar \), so this becomes: \( \frac{1}{4} (1 + \frac{1}{\hbar} S^x) (P_+ 4^{(1)}) = P_+ 4^{(1)} \).

Note: \( P_+ 4^{(1)} \) is an eigenstate of \( S^x_\parallel \), but \( 4^{(1)} \) itself is not!

Alternatively: \( P_+ + P_+ 4^{(1)} = \frac{1}{4} (1 + \frac{1}{\hbar} S^x_\parallel + \frac{1}{\hbar} S^z_\parallel) 4^{(1)} \)

Now \( S^x_\parallel \) is the square of the spin component along \( \parallel \).

You should have learned in QM, or from observation of spin operators, that any component of spin, e.g. \( S_x, S_y, S_z \), when squared, reduces to a constant times the identity, e.g. \( S_x^2 = \frac{\hbar^2}{4} I \), \( S_y^2 = \frac{\hbar^2}{4} I \), \( S_z^2 = \frac{\hbar^2}{4} I \).

Thus \( P_+ P_+ 4^{(1)} = \frac{1}{4} (1 + \frac{1}{\hbar} S^x + \frac{1}{\hbar} S^z) 4^{(1)} = \frac{1}{4} (1 + \frac{2}{\hbar} S^x_\parallel) 4^{(1)} = P_+ 4^{(1)} \)

b) \( (I, P_+, P_-) \) do not form a group since there is no inverse for \( P_+ \) or \( P_- \) in this set, and \( P_+ P_- = 0 \) is not part of the set.
6. Construct a gauge theory for scalar fields invariant under a local $SO(2)$. For the following you do **not** need to show your work, but you must be very clear in notating your answers.

a) Write down a Lagrangian for this theory, specifying the form of the transformation on the matter fields.

b) Promote this to a local symmetry by writing down a covariant derivative and transformation rule for the gauge field(s).

c) Write down gauge invariant kinetic term(s) for the gauge field(s). You must include the explicit form of $F_{\mu\nu}$.

d) Cheers!

e) **(Optional challenge question)** Would it be possible to do this starting with spinor fields?

\[ L = \frac{i}{2} \partial_\mu \phi^\dagger \partial^\mu \phi + \left( \frac{m}{4\pi^2} \right)^2 \phi^\dagger \phi \]

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \phi' = U \phi \quad \Rightarrow \quad \phi^\dagger \rightarrow \phi'^\dagger = (U \phi)^\dagger = \phi U^\dagger \]

\[ \text{Element of } SO(2) \text{ which is } \text{abelian} \]

\[ \text{so everything will commute!!} \]

\[ \partial_m \rightarrow \partial_m = \partial_m + A_m \]

We want: \[ \partial_m \phi \rightarrow \partial'_m \phi' = U \partial_m \phi \]

So we need: \[ \partial'_m \phi' = (\partial_m + A'_m) U \phi \]

\[ = \partial_m (U \phi) + U \partial_m \phi + A'_m U \phi \]

which we want to be \[ U (\partial_m \phi + A_m \phi) \]

This requires: \[ A_m \rightarrow A'_m = A_m - \partial_m (U \phi) U^{-1} \]

\[ \text{Then add } -\frac{i}{16\pi^2} F_{\mu\nu} F^{\mu\nu} \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ \text{This transpose is in color space} \]

\[ T^I T^{J} \delta_{IJ} \]

\[ \text{an } \text{SO}(4) \text{ matrix} \]

\[ \hat{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \rightarrow \hat{\gamma}' = U \hat{\gamma} U^\dagger \]

\[ \gamma_4 \]
7. Consider a potential for a three component Higgs field \((\varphi_1, \varphi_2, \varphi_3)\) of the form:

\[
U(\varphi_1, \varphi_2, \varphi_3) = -\frac{\mu^2}{2} (\varphi_1^2 + \varphi_2^2 + \varphi_3^2) + \frac{\lambda}{4} (\varphi_1^2 + \varphi_2^2 + \varphi_3^2)^2
\]

a) If this theory is expanded about the solution \(\varphi_1 = \frac{1}{\sqrt{3}} \lambda, \varphi_2 = \frac{1}{\sqrt{3}} \lambda, \varphi_3 = \frac{1}{\sqrt{3}} \lambda, A_\mu = 0\), with fluctuations \(\eta, \beta, \gamma\) respectively, what linear combination(s) of \(\eta, \beta, \gamma\) would correspond to the Higgs boson(s) and what would the mass(es) of the Higgs boson(s) be?

b) (Optional challenge question) What symmetry breaking does this Higgs potential facilitate? How many gauge bosons get mass?

a) This potential is spherically symmetric and the solution given is the point:

\[
\Phi_3 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\]

The Higgs boson is always the "radial" fluctuation so the combination \(\frac{1}{\sqrt{3}} \eta + \frac{1}{\sqrt{3}} \beta + \frac{1}{\sqrt{3}} \gamma\) would correspond to the single Higgs boson. The mass for potentials with coefficients \(\frac{m^2}{4}\) and \(\frac{\lambda}{4}\) always comes out to be \(M_H = \frac{\sqrt{2} \lambda}{\sqrt{3}}\).

b) The original potential had an SO(3) symmetry acting on \(\Phi = (\Phi_1, \Phi_2, \Phi_3)\) and preserving \(\Phi_1^2 + \Phi_2^2 + \Phi_3^2\), i.e. the length of a vector in 3D \(\Phi\)-space. After the Higgs takes a nonzero value, this is broken down to a subgroup of rotations in the plane orthogonal to the direction of the Higgs value. So we have \(SO(3) \rightarrow SO(2)\).

This is consistent since we would expect 2 Goldstone modes which would be eaten by 2 of the SO(3) generators leaving only 1 massless generator behind.