Rapidly Rotating Fermi Gases

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We show that the density profile of a Fermi gas in rapidly rotating potential will develop prominent features reflecting the underlying Landau-level–like energy spectrum. Depending on the aspect ratio of the trap, these features can be a sequence of ellipsoidal volumes or a sequence of quantized steps.

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Currently, there is an intense effort to cool trapped Fermi gases down to the degenerate limit. Recent experiments at JILA on $^{40}$K have reached one-half of their Fermi temperature [1]. One of the motivations of cooling fermions is to reveal their possible superfluid ground states, which can be quite novel in the case of multicomponent fermion systems [2]. Current theoretical estimates, however, indicate that the interactions between different spin states of $^{40}$K are all positive [3], implying a normal instead of superfluid ground state. The absence of superfluid ground states, however, does not mean that the system cannot have novel macroscopic quantum phenomena. Quantum Hall effect is an excellent example. In a strong magnetic field, the energy levels of a two-dimensional electron system organize into highly degenerate Landau levels, leading to a whole host of dramatic effects. In this paper, we show that similar organizations will take place in a fast rotating three-dimensional trapped Fermi gas, leading to many macroscopic quantum phenomena.

For neutral atoms in rotating harmonic traps, we shall see that Landau-level–like energy spectrum will appear when the rotation frequency $\Omega$ approaches the transverse confining frequency $\omega_{\perp}$. This “fast rotating” limit might appear hard to achieve as the system is at the verge of flying apart due to centrifugal instability [4]. Such instability, however, can be prevented by imposing an additional repulsive potential which dominates over the centrifugal force beyond a certain radius. With centrifugal instability eliminated, the Landau-like levels will show up in many ways. We will see that for cylindrical harmonic traps with $\omega_z \ll \omega_{\perp}$, where $\omega_{\perp}$ and $\omega_z$ are the transverse and longitudinal trapping frequencies, the density is a sum of one-dimensional–like density distributions residing in different “Landau volumes.” For traps with $\omega_z$ comparable or smaller than $\omega_{\perp}$, the density consists of a set of disks along $z$, each of which is made up of a sequence of density steps quantized in units of $M\omega_{\perp}/(\pi \hbar)$.

2D case: We first consider 2D rotating Fermi gases in harmonic potentials since they illustrate the basic physics. The Hamiltonian in the rotating frame is

$$H = \Omega L_z + \frac{1}{2M} p_\perp^2 + \frac{1}{2} M \omega_{\perp}^2 r^2 - \Omega \hat{z} \cdot \mathbf{r} \times \mathbf{p}_\perp,$$

(1)

where $\mathbf{p}_\perp = (p_x, p_y)$ and $\mathbf{r} = (x, y)$. The eigenfunctions and eigenvalues of Eq. (1) are

$$u_{n,m}(r, \theta) = \frac{e^{i\theta/2} \delta_m^0 \delta_n^0 e^{-|w|^2}}{\sqrt{\pi a_{\perp}^2 n! m!}},$$

(2)

$$\epsilon_{n,m} = \hbar (\omega_{\perp} + \Omega)n + \hbar(\omega_{\perp} - \Omega)m + \omega_{\perp},$$

(3)

where $n, m$ are non-negative integers $0, 1, 2, \ldots$, $w = (x + iy)/a_{\perp}$, $a_{\perp} = \sqrt{\hbar/M \omega_{\perp}}$, and $\delta_n^0 = (\delta_n \pm i \delta_n)$. To derive Eqs. (2) and (3), we note that Eq. (1) can be written as $\mathbf{H}\Omega/2M + (\omega_{\perp} - \Omega) L_z$ with $\mathbf{H} = \mathbf{p}_\perp - M \omega_{\perp} \hat{z} \times \mathbf{r}$, which is precisely the canonical momenta of an electron in a magnetic field $\mathbf{B}$ in the symmetric gauge, with $eB/Mc = 2\omega_{\perp}$. The eigenfunctions of $\mathbf{H}\Omega/2M$ are those in Eq. (2) [5], with eigenvalues $\epsilon_{n,m}^0 = \hbar (2n + 1)$, where $n$ is the Landau-level index, and $m$ is an “angular momentum” index labeling the degeneracy in each level. Since $L_z u_{n,m} = \hbar(n - m)u_{n,m}$, Eq. (2) is also an eigenstate of Eq. (1) with eigenvalues Eq. (3). Note that the function $u_{n,m}$ in Eq. (2) peaks at

$$r_{n,m} \equiv \langle r^2 \rangle_{n,m} = a_{\perp}^2(n + m + 1),$$

(4)

and decays away as a Gaussian over a distance $a_{\perp}$.

Equation (3) shows that the system is unbounded when $\Omega > \omega_{\perp}$ unless an additional repulsive potential $V_{wall}(r)$ (say, introduced by an additional optical trap) is present. We shall in particular consider potentials $V_{wall}(r)$ which are zero for $r < R$ but become strongly repulsive for $r > R$, with $R \gg a_{\perp}$. The specific form of $V_{wall}$ is not important for the key features discussed below, as long as it is smooth over length scale $a_{\perp}$. The condition $R \gg a_{\perp}$, however, allows us to fit many $m$ states inside $r < R$ and is a necessary feature for many effects discussed below. Since $V_{wall}(r)$ is cylindrically symmetric, the eigenstates are still labeled by quantum numbers $(n, m)$. For states originally with $r_{n,m} < R$, Eqs. (2) and (3) remain valid because $V_{wall} = 0$ for $r < R$. For states $(n, m)$ originally peaked beyond $R$, their energies increase rapidly because $V_{wall}$ is strongly repulsive. (For $^{40}$K in a tight trap $\omega_{\perp} = 4000$ and $10^3$ Hz, we have $a_{\perp} = 2.5 \times 10^{-5}$ and $5 \times 10^{-6}$ cm, respectively. The condition $R \gg a_{\perp}$ is satisfied for $R > 5 \times 10^{-4}$ cm.)
Let us first consider the case $\Omega < \omega_{\perp}$ with $V_{\text{wall}} = 0$. The density in the ground state is $\rho(r) = \sum_{n,m} |u_{n,m}(r)|^2 \Theta(\mu - \epsilon_{n,m})$, where $\Theta(x) = 1$ or 0 if $x > 0$ or $< 0$, and $\mu$ is the chemical potential related to the particle number $N$ as $N = \sum_{n,m} \Theta(\mu - \epsilon_{n,m})$.

We can write $\rho(r) = \sum_{n=0}^{\infty} \rho_n(r; m_n^*)$, where $\rho_n(r; L)$ is density contribution of the $n$th Landau level with angular momentum states filled up to $m = L$;

$$\rho_n(r; L) = \sum_{m=0}^{L} |u_{n,m}(r)|^2 \Theta(\mu - \epsilon_{n,m}),$$

$m_n^*$ is the highest angular momentum state in the $n$th Landau level with energy less than $\mu$, and $n^*$ is the highest Landau level below $\mu$,

$$m_n^* = \text{Int}[\frac{\mu}{2\hbar} + \frac{\Omega}{2}, \frac{\omega_{\perp}}{2\hbar}],$$

$n^* = \text{Int}[\frac{\mu}{2\hbar} + \frac{\Omega}{2}, \frac{\omega_{\perp}}{2\hbar}],$ where $\text{Int}[x]$ denotes the integer part of $x$, and $x$ is understood to be positive. Since $(n, m_n^*)$ is the state in $\rho_n$ farthest from the origin, its peak location $(r_n = r_{n,m_n^*} = a_{\perp} \sqrt{m_n^* + 1})$ gives the size of $\rho_n$. When $\Omega$ is very close to $\omega_{\perp}$, we have $m_n^* \gg 1$ and

$$r_n^2 = \frac{\mu - \hbar \Omega}{M \omega_{\perp} (\omega_{\perp} - \Omega)}.$$  

Note that the difference in area between successive Landau disks is a constant

$$\pi(r_n^2 - r_{n-1}^2) = \left(\pi a_{\perp}^2 \right) \left(\frac{2\Omega}{\omega_{\perp} - \Omega}\right).$$

By using Eq. (2), it is straightforward to show that

$$\rho_0(r; m_0^*) = \frac{1}{2\pi a_{\perp}^2} \left[1 - \text{erf}\left(\frac{s}{\sqrt{2}\omega_{\perp}}\right) (1 + \ldots)\right],$$

where $s = (r/a_{\perp})^2 - m_0^*$, $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-z^2} dz$, and the term $\ldots$ in Eq. (9) is of order $(12m_0^{-1})$ and smaller. The densities $\rho_0(r)$ and the higher Landau levels ($n > 0$) can be generated from $\rho_0$ as [6] $\rho_n(r; m_n^*) = a_{\perp}^2 \text{Int}[\frac{\mu}{2\hbar} + \frac{\Omega}{2}, \frac{\omega_{\perp}}{2\hbar}]$. From the properties of $\text{erf}(x)$, it is clear that $\rho_0$ is a constant $1/(\pi a_{\perp}^2)$ within a disk of radius $r_0 = \sqrt{m_0^* + 1} a_{\perp}$ and has an edge of width $\Delta_0 = \frac{3}{2\pi a_{\perp}}[7]$. If $m_0^* > 1$, $\Delta_0 < r_0$, and $\rho_0$ can be approximated as a step function on scales larger than $\Delta_0$. Likewise, $\rho_0$ can be approximated as a step function with somewhat larger width within the same approximation. Thus, when $m_n^* > 1$, we have

$$\rho_n(r) = \left(\pi a_{\perp}^2\right)^{-1} \Theta(r_n^2 - r^2).$$

Equations (10) and (8) then imply $\rho(r) = \frac{M\omega_{\perp}}{\pi \hbar} I(r)$, with

$$I(r) = \sum_n \Theta[\mu - M \omega_{\perp} (\omega_{\perp} - \Omega)r^2 + \hbar \Omega (2n + 1)].$$

By using the identity $\text{Int}[x + 1] = \sum_n \Theta[\alpha(x - n)]$, for all $x > 0$ and $\alpha > 0$, $\rho(r)$ can be simplified to

$$\rho(r) = \frac{M\omega_{\perp}}{\pi \hbar} \text{Int}\left[\frac{\mu - M \omega_{\perp} (\omega_{\perp} - \Omega)r^2 + \hbar \Omega}{2\hbar \Omega}\right].$$

It is, however, instructive to rederive Eq. (11) in a way generalizable to arbitrary potentials. We rewrite $\rho$ as

$$H - \Omega L_z = \left(\frac{\rho}{M} - \frac{\hbar}{2M} \hat{\mathbf{k}} \times \mathbf{r}\right)^2 + \frac{1}{2} M (\omega_{\perp}^2 - \Omega^2) r^2.$$  

The first term gives a set of $2\hbar$ Landau levels with spacing $2\hbar\Omega$, each of which contributes $(\pi a_{\perp}^2)^{-1}$ to the density, where $A^2 = \hbar/(M\Omega)$. If the second term in Eq. (12) is absent, the density is given by $\rho = I/(\pi a_{\perp}^2)$, where $I$ is the total number of Landau levels below the chemical potential, $I = \text{Int}[\frac{\mu + \hbar \Omega}{2\hbar \Omega}]$. When $\Omega$ is close to $\omega_{\perp}$, the second term in Eq. (12) is slowly varying over the scale of $A \sim a_{\perp}$ and can be absorbed in the chemical potential. The density profile within local density approximation (LDA) is then

$$\rho(r) = \frac{M\omega_{\perp}}{\pi \hbar} I(r), \quad I(r) = \text{Int}\left[\frac{\mu(r) + \hbar \Omega}{2\hbar \Omega}\right].$$

where $\mu(r) = \mu - \frac{1}{2} M (\omega_{\perp}^2 - \Omega^2) r^2$. Clearly, Eq. (13) is equivalent to (11) up to correction $(1 - \Omega/\omega_{\perp}) \ll 1$ as $\Omega \rightarrow \omega_{\perp}$. Equation (13), once established, is easily generalized to other potentials. In the presence of $V_{\text{wall}}$, simply replaces $\mu(r)$ in Eq. (13) by

$$\mu(r) = \mu - \frac{1}{2} M (\omega_{\perp}^2 - \Omega^2) r^2 - V_{\text{wall}}(r).$$

Equations (13) and (14) constitute the LDA solution for the 2D rotating Fermi gas for both $\Omega < \omega_{\perp}$ and $\Omega > \omega_{\perp}$. The schematics of LDA is shown in Figs. 1(a) and 1(b).

To understand the validity of LDA [Eq. (13)], we calculated the density numerically using Eq. (2). The result for a system of 2000 fermions at $\Omega/\omega_{\perp} = 0.996$ is shown in Fig. 2(a). The system exhibits a sequence of quantized steps at locations well described by LDA. The evolution of the density within the range $0.99 < \Omega/\omega_{\perp} < 1$ of this fermion system is shown in Fig. 2(b). As $\Omega$ decreases, more Landau levels are populated while the steps near the surfaces are closer together [as expected from Eqs. (6) and (8)], yet the step structures remain discernable and correctly described by LDA [the LDA construction is not displayed, so as to keep Fig. 2(b) readable]. The behaviors of the densities at lower frequency $0.98 < \Omega/\omega_{\perp} < 1$ are shown in Figs. 3(a) and 3(b) for a system of 1000 fermions. At the lowest frequency displayed [Fig. 3(a)], the step structure near the surface is completely smeared out by the spread of the edges. Nevertheless, the density of the innermost plateau remained quantized, with a size correctly described by LDA. This core of quantized density (or “quantized core” for short) is clear evidence for Landau levels. Our studies show that, for about 2000 particles, Landau levels will show up as a sequence.
of discernable steps only when $0.99 < \Omega / \omega_\perp < 1$, which is achievable with the current capability to control frequencies, especially for large $\omega_\perp$. For lower frequencies, the existence of Landau levels can only be revealed through the presence of a quantized core, which shrinks in size as $\Omega$ decreases. On the other hand, the LDA in Fig. 1(b) shows that, by introducing an additional potential $V_{\text{wall}}$, Landau levels (in the form of a sequence of steps near the center or a quantized core) can still be revealed at frequencies farther beyond $\omega_\perp$, even though the steps near the surface are smeared out.

3D case: For a 3D harmonic trap, Eq. (1) acquires terms $\rho_z^2/2M + \frac{1}{2}M \omega_z^2 z^2$, which give rise to harmonic oscillator eigenfunctions $f_{n_z}(z)$ with eigenvalue $\epsilon_{n_z} = \hbar \omega_z (n_z + \frac{1}{2})$. The density is $\rho(r, z) = \sum_{n, m} |f_{n_z}(z)|^2 |u_{n_z m}(r)|^2 \times \Theta(\mu - \epsilon_{n_z m})$, where $\epsilon_{n_z m} = \epsilon_{n_z} + \epsilon_{m-1,0}$.

When $\Omega / \omega_\perp \approx 1$, we first perform the $m$ sum because it produces the smoothest change in the energy. By repeating the steps leading to Eq. (10), we have

$$\rho(r, z) = \sum_{n, m} \frac{|f_{n_z}(z)|^2}{\pi a_\perp^2} \Theta(\mu - \hbar \Omega (2n + 1) - \epsilon_{n_z}),$$

(15)

where $\mu(r) = \mu - M \omega_z (\omega_\perp - \Omega)^2 M (\omega_\perp^2 - \Omega^2) r^2$.

The order of summation of the remaining integers $n_z$ and $n$ depends on the relative strength between $\omega_z$ and $\omega_\perp$. When $\omega_z \ll \omega_\perp$, we first sum $n_z$. To do that, we note that the density of a 1D Fermi gas is $\rho_{1D} = \frac{\sqrt{2 \mu}}{\pi \hbar}$, where $\mu$ is the chemical potential. In a harmonic trap, the density within LDA is obtained by changing $\mu$ to $\mu(z) = \mu - \frac{1}{2} M \omega_z^2 z^2$. We then have $\rho_{1D}(z) = \sum_{n_z} |f_{n_z}(z)|^2 \Theta(\mu - \epsilon_{n_z}) = \frac{\sqrt{2 \mu(z)}}{\pi \hbar}$, which turns
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To demonstrate the validity of the LDA equation (16),

\[ \rho(r) = \frac{1}{\pi^2 \frac{\bar{a}_z}{\bar{a}_z}} \times \sum_n \sqrt{[\mu(r, z) - \hbar \Omega(2n + 1)]/\frac{1}{2}\hbar \omega_z} \] (16)

where \( \bar{a}_z = \sqrt{\hbar/M \omega_z} \), and \( \mu(r, z) = \mu - \frac{M}{\pi} (\omega_z^2 - \Omega^2)r^2 - \frac{\hbar}{\pi} \omega_z^2 z^2 - V_{\text{wall}} \). We have included \( V_{\text{wall}} \) in \( \mu \) for the more general situation as in the 2D case. Equation (16) shows that \( \rho(r) \) is a sum of 1D densities (labeled by “\( n \)”), each of which is distributed over in a “Landau volume” bounded by the “Landau surface” \( \mu(r, z) = \hbar \Omega(2n + 1) \). When \( V_{\text{wall}} = 0 \), \( \Omega < \omega_z \) the Landau surfaces are ellipsoidal surfaces. It is easy to verify that the surface areas \( A_n \) for successive ellipsoids differ by a constant \( A_{n-1} - A_n = \frac{\hbar}{M \pi} (\frac{16\pi\Omega^2}{\omega_z - \Omega^2}) \). When \( \Omega \geq \omega_z \), centrifugal instability against harmonic confinement sets in and stability can only be established by \( V_{\text{wall}} \).

To demonstrate the validity of the LDA equation (16), we have evaluated the density numerically for a system of 2000 fermions for \( \omega_z/\omega_\perp = 0.2 \) at \( \Omega/\omega_\perp = 0.99 \). The results are shown in Fig. 4. It shows that the LDA (dotted line) is a good approximation. The Landau volumes can be clearly identified by the change of slope in the density. The appearance of a plateau at the center is because \( \omega_z/\omega_\perp \) is only 0.2, revealing the 2D feature of the \( n \) levels. For smaller ratios of \( \omega_z/\omega_\perp \), the plateau disappears and the LDA expression [Eq. (16)] is achieved.

When \( \omega_z > \omega_\perp \), summation of \( n \) in Eq. (15) gives

\[ \rho(r) = \frac{M \omega_\perp}{\pi \hbar} \sum_{n_z} |f_{n_z}(z)|^2 \int \left[ \frac{\mu(r) - \hbar \omega_z(n_z + \frac{1}{2}) + \hbar \Omega}{2\hbar \Omega} \right] \] (17)

where \( \mu(r) = \mu - \frac{1}{2}M(\omega_z^2 - \Omega^2)r^2 - V_{\text{wall}} \). In this limit, the density consists of a sequence of disks (labeled by \( n_z \)) in the \( z \) direction. Each disk \( |f_{n_z}(z)|^2 \) consists of a sequence of density steps in the \( xy \) plane reflecting the number of filled Landau levels. The behavior of the density within each disk in the \( xy \) plane is identical to the 2D case discussed previously. Finally, we note that, as the temperature increases, the Landau-level structure near the surface will first melt away, and the melting will proceed toward the center. The temperature below which the Landau-level effect begins to appear is \( T = 2\hbar \omega_\perp/k_B \), which is \( 3.8 \times 10^{-7} \) and \( 9.6 \times 10^{-6} \) K for \( \omega_\perp = 4000 \) and \( 10^5 \) Hz, respectively, a temperature range achievable in current experiments [8].

Thus far, we have discussed only the effect of Landau levels on the density profiles of fast rotating Fermi gases. If the development of the quantum Hall effect in the past decade is a guide, one expects many more novel phenomena in Fermi gases in the fast rotating regime.

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[4] In the recent experiment of F. Chevy, K. W. Madison, and J. Dalibard (cond-mat/0005221) on rotating Bose gas, the highest rotational frequency used is quite close to \( \omega_z \).
[6] Note that \( \rho_0(r, L) = \sum_{m=0}^{L} |e^{iLm}\hbar^2/w^2|^{1/2} \), where \( \rho_{nm, k} = \rho_{nm, k}^{(1)} + \rho_{nm, k}^{(2)} \).}

\[ \rho_0(r, L) = \frac{1}{(\pi n!)^2} \sum_{m=0}^{L} |e^{iLm}\hbar^2/w^2|^{1/2} \int e^{-x^2} e^{-y^2} \left[ \frac{1}{2\pi} \sum_{m=0}^{L} |e^{iLm}\hbar^2/w^2|^{1/2} \right] \] (18)

where \( \Delta = \frac{\hbar^2}{2m^2} \bar{a}_z \). At higher temperatures, the density profile is that of a Boltzmann gas, \( \rho = e^{-M(\omega_\perp^2 - \Omega^2)r^2}/(2\hbar \Omega) \).