Figure 1: A sequence $x[n]$ and its amplitude spectrum. The sequence $x[n]$ is not aliased and consists of 126 samples of a continuous signal $x_c(t)$. The periodic fluctuations are noise; the interesting signal is about -40 dB down.

**Question 1** ................................................................. (10 points)

(a) [2 points] What is the time sampling interval $T$, in seconds?

(b) [2 points] What is the Nyquist frequency, in Hz?

(c) [2 points] What is the frequency (in Hz) of the periodic fluctuations in $x[n]$.

(d) [2 points] Label the frequency axis below the amplitude spectrum.

(e) [2 points] As noted above, the periodic fluctuations are noise; the signal of interest is about -40 dB down. What does “-40 dB down” mean? Specifically, what is the ratio of signal amplitude to noise amplitude?
Assume that you are given a discrete Fourier transform $X[k]$ of the sequence $x[n]$ of Figure 1. Assume also that a fast Fourier transform (FFT) was used, and that the number of samples after padding $x[n]$ with zeros was $N = 250$.

(a) [2 points] If only positive frequencies $\omega$ are sampled, how many complex values are provided in the array $X[k]$?

(b) [2 points] For what sample indices $k$ are the imaginary parts of $X[k]$ zero?

(c) [2 points] What is the frequency sampling interval $\Delta \omega$, in radians/sample?

(d) [2 points] What is the frequency sampling interval $\Delta F$, in Hz?

(e) [2 points] Imagine a simple filter that zeros amplitudes for frequencies between 1 and 3 Hz. To implement this filter, for what range of sample indices $k$ would you zero $X[k]$?
Question 3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (12 points)
Design a causal notch filter to attenuate the periodic fluctuations in $x[n]$ of Figure 1, as follows:

(a) [2 points] What is the frequency to be attenuated, in radians/sample?

(b) [2 points] Sketch the locations of filter poles and zeros in the complex $z$-plane.

(c) [2 points] What is the system function $H(z)$ for your filter? (Include the region of convergence.)

(d) [2 points] Modify your system function $H(z)$ so that your filter does nothing at frequency zero (DC).

(e) [2 points] Write a constant-coefficient difference equation relating filter output $y[n]$ to input $x[n]$.

(f) [2 points] Write the main loop of a computer program that implements your filter.
Question 4 ................................................................. (12 points)

Consider an LTI system with the following frequency response

\[ H(\omega) = \begin{cases} 1, & \text{if } |\omega| \leq \pi/2, \\ 0, & \text{otherwise}. \end{cases} \]

(a) [2 points] Sketch this frequency response \( H(\omega) \) for frequencies \( \omega \) in the range \(-\pi \leq \omega \leq \pi\).

(b) [4 points] What is the impulse response \( h[n] \) of this system?

(c) [2 points] Sketch the impulse response \( h[n] \) of this system.

(d) [2 points] Suppose the sequence \( x[n] \) of Figure 1 is input to this system to obtain an output sequence \( y[n] \). Using the amplitude spectrum in Figure 1 as a guide, sketch the amplitude spectrum of the output sequence \( y[n] \).

(e) [2 points] Such a system might be used prior to subsampling the sequence \( y[n] \). Specifically, we might use it before computing \( z[n] = y[2n] \). Why?
Question 5 .......................................................... (11 points)

Consider three moving-average filters with system functions:

\[ H_1(z) = \frac{1}{3}(1 + z^{-1} + z^{-2}) \]
\[ H_2(z) = \frac{1}{3}(z^2 + z + 1) \]
\[ H_3(z) = H_1(z) H_2(z) \]

(a) [4 points] Sketch the impulse responses of all three systems.

(b) [1 point] For what frequency \( \omega \) between 0 and \( \pi \) is the amplitude response \( A_1(\omega) \) of the filter \( H_1 \) zero?

(c) [1 point] For what frequency \( \omega \) between 0 and \( \pi \) is the amplitude response \( A_2(\omega) \) of the filter \( H_2 \) zero?

(d) [1 point] For what frequency \( \omega \) between 0 and \( \pi \) is the amplitude response \( A_3(\omega) \) of the filter \( H_3 \) zero?

(e) [2 points] Give two reasons why the filter \( H_3 \) is a better moving-average filter than either \( H_1 \) or \( H_2 \).

(f) [2 points] Which of these filters are causal? Which are stable?