God doesn't play dice -- Albert Einstein

Einstein's famous quotation was not about his speculations concerning the gambling propensities of God, but rather an expression of his dissatisfaction with the apparently probabilistic description of nature embodied by the quantum theory.

Suppose God did favor games of chance, and that he opened two casinos: the Classical Casino, and the Quantum Casino. In both casinos, craps is the game played, where two independent dice are rolled, and (for simplicity here), a winning roll has a total of seven, while a losing roll has a total of two ("snake eyes" or "craps").

In the Classical Casino, the laws of probability are the classical ones, while in the Quantum Casino, the die are indistinguishable, and for this particular homework, they obey Fermi statistics.

Warm up before entering the casinos:

1) For the Classical Casino, how many possible different rolls (states of the combined system of two die) are there, if the dice are distinguishable? How many possible rolls are there for the two Fermionic dice?

In classical probability theory, two independent distinguishable dice will each have 6 possible faces, and so there are a total of 36= 6 x 6 combinations between the two. (There are 36 ordered pairs. The order distinguishes one die from the other.)

	1	**2**	**3**	**4**	**5**	**6**
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

However, for indistinguishable dice, a "3 , 4" is not considered a different roll from "4 , 3", and so only half the off-diagonal cells and the diagonals are considered to be different rolls (states):

	1	**2**	**3**	**4**	**5**	**6**
1	1,1	1,2	1,3	1,4	1,5	1,6
2		2,2	2,3	2,4	2,5	2,6
3			3,3	3,4	3,5	3,6
4				4,4	4,5	4,6
5					5,5	5,6
6						6,6

Thus there are only 21 distinct unordered pairs. Of these, however, the diagonal has zero weight for fermionic dice, because they cannot be in the same individual state. This leaves 15 possible rolls for fermionic dice (15 unordered pairs of two *different* numbers). Note: bosonic dice would have no such restriction, and have 21 possible states of the two dice system.

2) What is the probability of winning and losing in the Classical Casino? In the Quantum Casino?

So the fermionic dice have a table which looks like:

	1	**2**	**3**	**4**	**5**	**6**
1		1,2	1,3	1,4	1,5	1,6
2			2,3	2,4	2,5	2,6
3				3,4	3,5	3,6
4					4,5	4,6
5						5,6
6						

Winning, corresponds to a "7" total, which can only occur for (3,4), (2,5) and (1,6), or a probabiliity of wining with fermionic dice of 3/15 = 1/5. The probability of losing with fermionic dice is *zero*, because snake eyes (1,1) is not allowed for fermions.

In contrast, for classical dice, there are 6 ways to make "7" : (1,6), (2,5), (3,4), (4,3), (5,2), (6,1),…corresponding to the diagonal at 45°….and so the probability of

winning with classical dice is 6/36 = 1/6. Less than the Quantum Casino! Moreover, there is a 1/36 chance of losing : rolling snake eyes (1,1).

Now, suppose that Lucifer has managed to substitute the dice in each casino with "loaded dice". In particular, each die has an off-center weight inside which causes the energy of the roll "one" to be lower than the average by $\Delta E = -\Delta(mgh)$, while a "six" has an energy which is higher than the average by $\Delta E = \Delta(mgh)$. (Take the zero of energy to be the energy of all the other rolls.)

In the first part of this homework dealing with fair dice, notice that we implicitly assumed all rolls (a square in the chart) had equal probability. One way of couching that in statistical mechanical terms is to assume that all the rolls had the same "energy" E, and thus all had the same Boltzmann weight exp (-E/k T). the partition function, Q, is just the number of rolls (either 36 for classical or 15 for fermions) times the same Boltzmann factor. Thus the probability is just exp(–E/k T) / Q = 1/36 or 1/15 for classical or fermionic, respectively.

Now, for loaded dice, the energies are not all equal, and thus neither are the Boltzmann factors. Let's make a chart first of the energies:

	1	**2**	**3**	**4**	**5**	**6**
1	$-2\Delta(mgh)$	$-\Delta(mgh)$	$-\Delta(mgh)$	$-\Delta(mgh)$	$-\Delta(mgh)$	0
2	$-\Delta(mgh)$	0	0	0	0	$\Delta(mgh)$
3	$-\Delta(mgh)$	0	0	0	0	$\Delta(mgh)$
4	$-\Delta(mgh)$	0	0	0	0	$\Delta(mgh)$
5	$-\Delta(mgh)$	0	0	0	0	$\Delta(mgh)$
6	0	$\Delta(mgh)$	$\Delta(mgh)$	$\Delta(mgh)$	$\Delta(mgh)$	$2\Delta(mgh)$

The Boltzmann factors are just the exponential of minus these energies / k T. For example, the Boltzmann factor for (1,2) = exp[Δ(mgh) / k T].

Questions:

1) What is the partition function for the two loaded dice in the Classical Casino? In the Quantum Casino?

For the Classical Casino, the entire 6 x 6 matrix of rolls are all considered different states, and so the partition function is just the sum of the Boltzmann factors for all 36 squares:

$$Q_{class} = \exp[2\,\Delta(mgh)/kT] + 8\exp[\Delta(mgh)/kT] + 18 + 8\exp[-\Delta(mgh)/kT] + \exp[-2\,\Delta(mgh)/kT]$$

The difference for the Fermionic Dice (still loaded) is that again two rolls which only differ in order are not considered to be different states. Thus, we only have the

same 15 states to consider as before, but now the sum of these 15 loaded Boltzmann factors are:
Q_{fermi} = 4 exp[Δ(mgh) / k T] + 7 + 4 exp[- Δ(mgh) / k T]

2) In which Casino are the odds better for winning? Does it matter whether the dice are "cold", $k_B T << \Delta$(mgh), or "hot" $k_B T >> \Delta$(mgh)? Calculate the odds in the two limits of T=0 and T = infinity.

As before, the odds of winning are the sum of the probabilities of the different winning rolls. For classical dice, it is the same 6 rolls as before, and their probabilities are each their Boltzmann factor divided by Q. For the 6 winning rolls, however, the energies are all zero, and so the Boltzmann factors are all one, thus:

Probability of winning in the Classical Casino with loaded dice = 6 / Q_{class}

$$= 6 / \{ \exp[2 \Delta(mgh) / k T] + 8 \exp[\Delta(mgh) / k T] + 18 + 8 \exp[- \Delta(mgh) / k T] + \exp[- 2 \Delta(mgh) / k T] \}$$

Similarly, the probability of winning in the Quantum Casino is = 3 / Q_{fermi}

$$= 3 / \{ 4 \exp[\Delta(mgh) / k T] + 7 + 4 \exp[- \Delta(mgh) / k T] \}$$

Notice that $Q_{fermi} < Q_{class} / 2$ (plug it in to verify), and so $3 / Q_{fermi} > 6 / Q_{class}$ regardless of the temperature. So even with loaded dice, it's better to play at the Quantum Casino. (Quick question: what are the chances of losing at the Quantum Casino with loaded dice?)

In the limit that the dice are "cold", $k_B T << \Delta$(mgh), or "hot" $k_B T >> \Delta$(mgh), we can take the limit of the above expressions for the probability. However, a simpler method is to realize that when the dice are "cold", the Boltzmann factor for the lowest level dominates over all the other levels. Hence, classically, the dice are always rolling snake eyes! Quantum mechanically, the dice are evenly distributed between the 4 degenerate states of the lowest level: (1,2), (1,3), (1,4), (1,5).

When the dice are "hot", all the Boltzmann factors go to 1, and then the probabilities are exactly what they were in the unloaded case (whether classical or quantum)!

There are only two ways to live your life. One is as though nothing is a miracle. The other is as though everything is a miracle. -- Albert Einstein