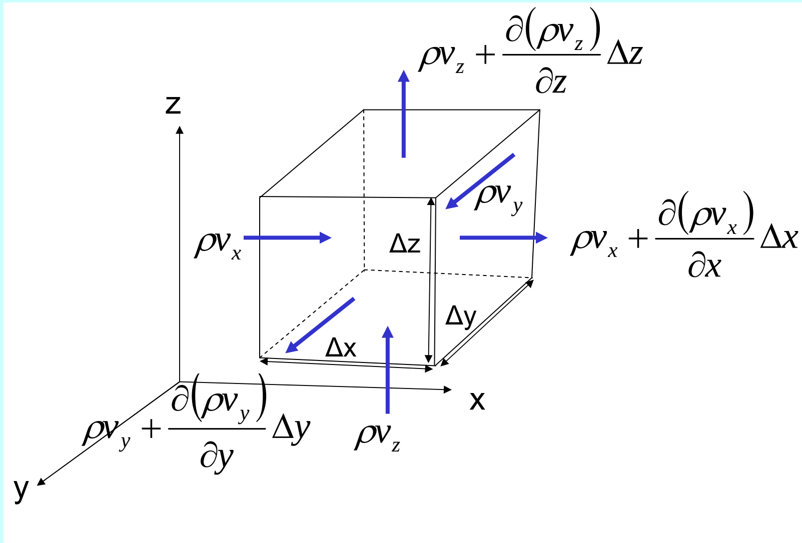
EQUATIONS OF GROUNDWATER FLOW

DARCY'S LAW + CONTINUITY EQUATION

ELEMENTAL CONTROL VOLUME:



INWARD FLOW OF MASS FLOW PER UNIT TIME

on the ΔyΔz faces

$$\rho \mathbf{v}_x \Delta y \Delta z - \left[\rho \mathbf{v}_x + \frac{\partial(\rho \mathbf{v}_x)}{\partial x}\Delta x\right]\Delta y \Delta z$$

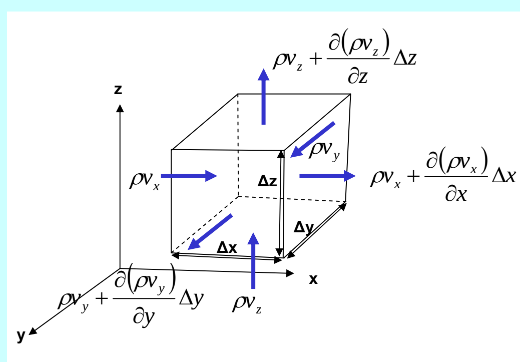
$$= -\left[\frac{\partial(\rho \mathbf{v}_x)}{\partial x}\right]\Delta x \Delta y \Delta z$$

similarly on the ΔxΔy faces

$$= -\left[\frac{\partial(\rho \mathbf{v}_z)}{\partial z}\right]\Delta z \Delta x \Delta y$$

similarly on the ΔxΔz faces

$$= -\left[\frac{\partial(\rho \mathbf{v}_y)}{\partial y}\right]\Delta y \Delta x \Delta z$$



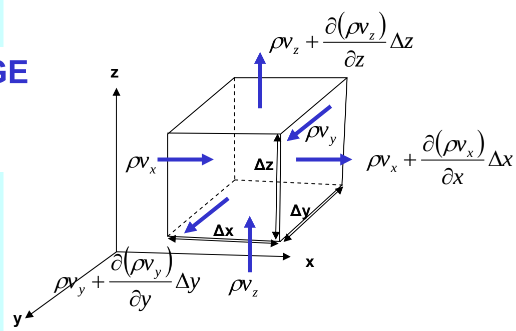
TOTAL MASS INFLOW PER UNIT TIME

$$= -\left[\frac{\partial(\rho v_x)}{\partial x}\right]\Delta x \Delta y \Delta z - \left[\frac{\partial(\rho v_y)}{\partial y}\right]\Delta y \Delta x \Delta z - \left[\frac{\partial(\rho v_z)}{\partial z}\right]\Delta z \Delta x \Delta y$$

Gathering terms and passing to differentials

$$= -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]dxdydz$$

MUST EQUAL THE CHANGE IN MASS STORAGE PER UNIT TIME

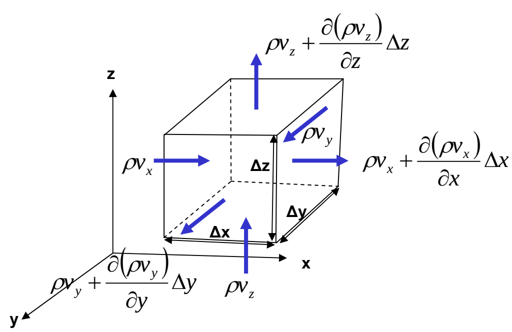


MASS OF WATER IN THE VOLUME ELEMENT

$$\Delta M = \rho \phi \Delta x \Delta y \Delta z$$

Mass changes with time as pressure changes

$$\frac{\partial M}{\partial t} = \rho(\alpha + \beta\phi)dxdydz\frac{\partial P}{\partial t}$$



$$P = P_o + \gamma h \quad \text{and} \quad \frac{\partial P}{\partial t} = \frac{\partial \gamma h}{\partial t}$$

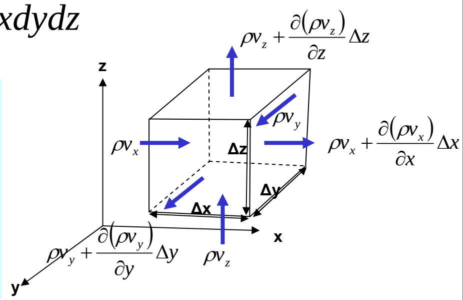
assume γ constant

So …. take gamma out of differential …
then equate MASS FLOW to CHANGE IN MASS WITH TIME

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial h}{\partial t}$$

equate MASS FLOW

$$-\left[\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}\right]dxdydz$$



to CHANGE in MASS with time

$$\rho(\alpha+\beta\phi)\,dxdydz\;\;\gamma\frac{\partial h}{\partial t}$$

$$-\rho\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]dxdydz=\rho(\alpha+\beta\phi)\,dxdydz\;\;\gamma\frac{\partial h}{\partial t}$$

cancel density and volume

$$-\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]=\gamma(\alpha+\beta\phi)\frac{\partial h}{\partial t}$$

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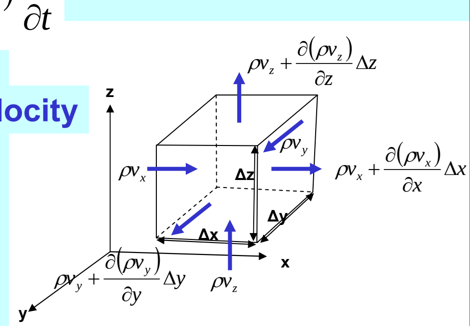


Substitute Darcy's Law for velocity

$$v=K\frac{\partial h}{\partial l}=K\frac{\partial h}{\partial x}$$

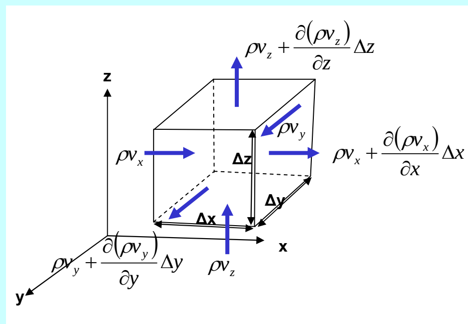
Basic flow equation for shallow ground water flow

$$\left[\mathbf{K}_x\frac{\partial^2\mathbf{h}}{\partial\mathbf{x}^2}+\mathbf{K}_y\frac{\partial^2\mathbf{h}}{\partial\mathbf{y}^2}+\mathbf{K}_z\frac{\partial^2\mathbf{h}}{\partial\mathbf{z}^2}\right]=\gamma(\alpha+\beta\phi)\frac{\partial\mathbf{h}}{\partial\mathbf{t}}$$

$$\left[\mathbf{K}_x \frac{\partial^2 \mathbf{h}}{\partial \mathbf{x}^2} + \mathbf{K}_y \frac{\partial^2 \mathbf{h}}{\partial \mathbf{y}^2} + \mathbf{K}_z \frac{\partial^2 \mathbf{h}}{\partial \mathbf{z}^2}\right] = \gamma(\alpha + \beta\phi)\frac{\partial \mathbf{h}}{\partial \mathbf{t}}$$

Substitute S_s for $\gamma(\alpha+\beta\phi)$
and
if homogeneous and isotropic, use constant K

$$K\left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right] = S_s \frac{\partial h}{\partial t}$$



$$K\left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right] = S_s \frac{\partial h}{\partial t}$$

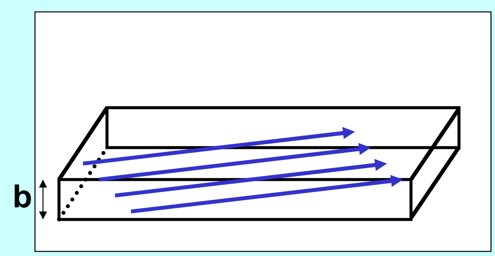
For 2D confined flow parallel to the formation:

Vertically average the properties and discard the z term:

$$Kb\left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right] = S_s b \frac{\partial h}{\partial t}$$

Recall definition of T and S

$$\mathbf{T}\left[\frac{\partial^2 \mathbf{h}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{h}}{\partial \mathbf{y}^2}\right] = \mathbf{S}\frac{\partial \mathbf{h}}{\partial \mathbf{t}}$$



For steady flow:

$$\left[\frac{\partial^2 \mathbf{h}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{h}}{\partial \mathbf{y}^2}\right] = \mathbf{0}$$

Laplace Equation

$$K\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}\right]=S_s\frac{\partial h}{\partial t}$$

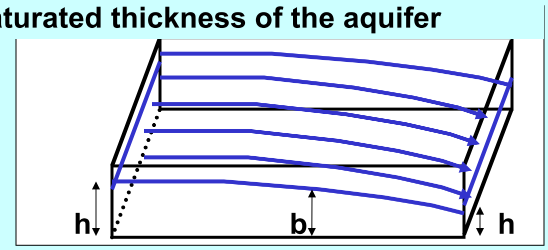
For 2D unconfined, flow thickness can change with time:

Boussinesq equation

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)=\frac{S_y}{K}\frac{\partial h}{\partial t}$$

is nonlinear, (i.e. Transmissivity varies with head so the properties are dependent on the solution). However, if the change in thickness of the aquifer is small we can replace the varying h (measured with the aquifer bottom as datum) with b, the average saturated thickness of the aquifer

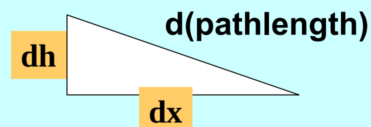
$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S_y}{Kb}\frac{\partial h}{\partial t}$$



For unconfined flow

Using head drop over x rather than the path length, l is referred to as the Dupuit Assumption:

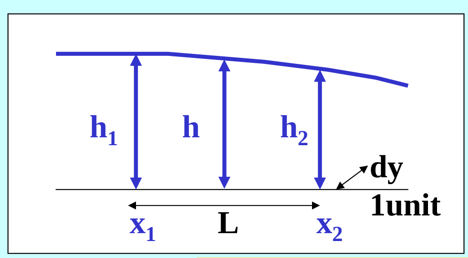
$$\mathbf{q=-Kh\frac{dh}{dx}}$$



$$\mathbf{qdx=-Khdh}$$

$$\int_0^L \mathbf{qdx}=-\mathbf{K}\int_{h_1}^{h_2}\mathbf{hdh}$$

$$\mathbf{qx}\Big|_0^L=-\mathbf{K}\frac{\mathbf{h}^2}{2}\Big|_{h_1}^{h_2}$$



$$\mathbf{qL=-K\left(\frac{h_2{}^2}{2}-\frac{h_1{}^2}{2}\right)}$$

or solving for q:

$$q=-K\frac{\left(\frac{h_2{}^2}{2}-\frac{h_1{}^2}{2}\right)}{L}$$

If we do the same for confined flow (linear), now aquifer thickness is a constant, b, rather than defined by h

$$q = -Kb\frac{dh}{dx}$$

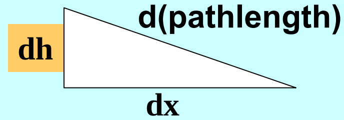
$$qdx = -Kbdh$$

$$\int_0^L qdx = -Kb\int_{h_1}^{h_2} dh$$

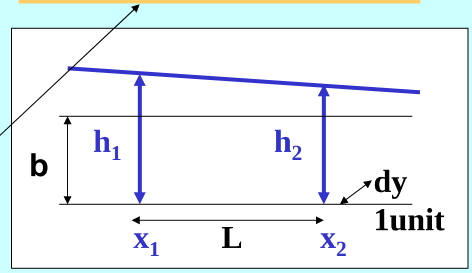
$$qx\Big|_0^L = -Kbh\Big|_{h_1}^{h_2}$$

$$qL = -Kb(h_2 - h_1)$$

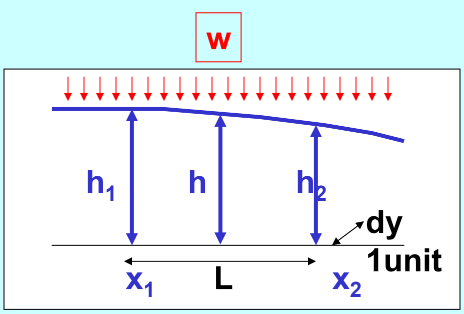
$$q = -Kb\frac{h_2 - h_1}{L}$$



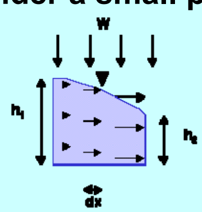
$$Q = KAi \quad \text{just Darcy's Law}$$



Extending the unconfined flow to a case with recharge:
Flow rate will not be the same at each x now



Consider a small prism:



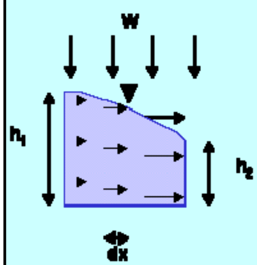
flow through the left face is:

$$q_x dy = -K\left(h\frac{\partial h}{\partial x}\right)_x dy$$

dy is into the screen
flow through the right face is:

$$q_{x+dx} dy = -K\left(h\frac{\partial h}{\partial x}\right)_{x+dx} dy$$

Notice the value of $\left(h\frac{\partial h}{\partial x}\right)$ **differs due to added recharge**



Difference in flow between faces equals inflow from the top:

$$(q_{x+dx} - q_x)dy = w\,dx\,dy = -K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dx\,dy$$

If w=0 this reduces to Laplace equation:

$$0 = -K\frac{\partial^2 h^2}{\partial x^2}dx\,dy$$

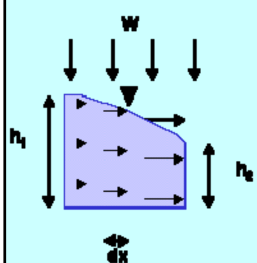
otherwise:

$$w = -K\frac{d^2h^2}{dx^2} \quad or \quad \frac{d^2h^2}{dx^2} = -\frac{w}{K}$$

integrating this yields:

$$h^2 = -\frac{wx^2}{K} + C_1 x + C_2$$

where C_1 and C_2 are constants of integration



$$h^2 = -\frac{wx^2}{K} + C_1 x + C_2$$

applying boundary conditions:

@ $x=0,\ h=h_1$ @ $x=L,\ h=h_2$

solving for constants:

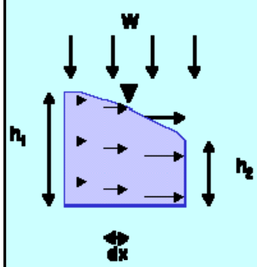
$$C_2 = h_1^2 \qquad C_1 = \frac{h_2^2 - h_1^2}{L} + \frac{wL}{K}$$

$$h_x^2 = h_1^2 - \frac{(h_1^2 - h_2^2)x}{L} + \frac{w}{K}(L-x)x$$

solving for h:

$$h_x = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)x}{L} + \frac{w}{K}(L-x)x}$$

To get q as a function of x differentiate with respect to x:



$$h_x^2 = h_1^2 - \frac{\left(h_1^2 - h_2^2\right)x}{L} + \frac{w}{K}(L-x)x$$

and because:

$$q_x = -Kh\left(\frac{dh}{dx}\right)$$

we have:

$$q_x = \frac{K\left(h_1^2 - h_2^2\right)}{2L} - w\left(\frac{L}{2} - x\right)$$

find the groundwater divide by setting q=0

$$d = \frac{L}{2} - \frac{K}{w}\frac{\left(h_1^2 - h_2^2\right)}{2L}$$

and maximum h by setting x=d

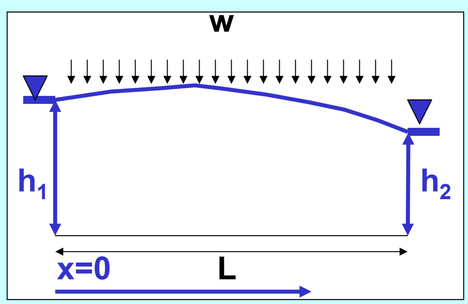
$$h_{max} = \sqrt{h_1^2 - \frac{\left(h_1^2 - h_2^2\right)d}{L} + \frac{w}{K}(L-d)d}$$

In summary, the unconfined flow case with recharge:

$$h_x = \sqrt{h_1^2 - \frac{\left(h_1^2 - h_2^2\right)x}{L} + \frac{w}{K}(L-x)x}$$

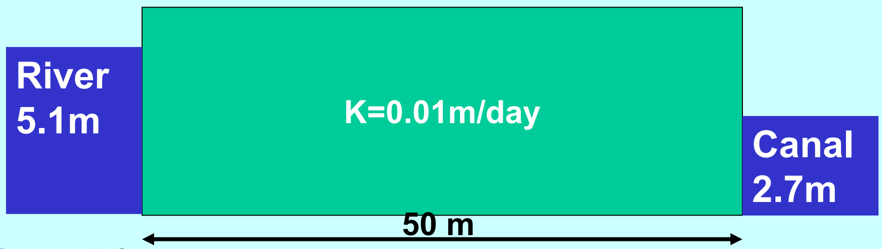
$$q_x = \frac{K\left(h_1^2 - h_2^2\right)}{2L} - w\left(\frac{L}{2} - x\right)$$

$$d = \frac{L}{2} - \frac{K}{w}\frac{h_1^2 - h_2^2}{2L}$$



$$h_{max} = \sqrt{h_1^2 - \frac{\left(h_1^2 - h_2^2\right)d}{L} + \frac{w}{K}(L-d)d}$$

A canal parallels a river 50 m to its west. The maximum ground surface elevation between them is 6 meters. Both the river and the canal fully penetrate an aquifer having a hydraulic conductivity of 0.01 m/day. Precipitation is 0.5 m/year, evapotranspiration is 0.4 m/year. The river is 5.1 m deep while the canal is 2.7 m deep.



Group 1 Calculate d, h_{max}, $h_{x=12.5}$, $h_{x=37.5}$, q at the river, and q at the canal. Sketch a diagram illustrating the shape of the water table and indicating the discharges.

Group 2 do the same except, $K = 1 \times 10^{-4}$ m/day

Group 3 do the same except, $K = 1 \times 10^{0}$ m/day

Group 4 do the same except, ET = 0.55 m/year and K = 0.01 m/day

Group 5 do the same except, ET = 0.5 m/year and K = 0.01 m/day